

GCSE HIGHER: HARDER PROBLEMS

Vector Ratios and Intersections

Worked Solutions

Q1

(a) $\overrightarrow{AB} = \frac{3}{7}\mathbf{m}$

The ratio 3:4 gives AB three of seven equal parts in AC .

$$\overrightarrow{BA} = -\frac{3}{7}\mathbf{m}$$

Reverse AB .

(b) $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$

Follow B to C to D .

$$\overrightarrow{BD} = \frac{4}{7}\mathbf{m} + \frac{2}{7}\mathbf{m} = \frac{6}{7}\mathbf{m}$$

Add the two parts of the route.

(c) $\overrightarrow{DA} = -(\mathbf{m} + \frac{2}{7}\mathbf{m}) = -\frac{9}{7}\mathbf{m}$

Reverse the whole route A to C to D .

Q2

(a) $\overrightarrow{AD} = -\mathbf{n}$

The given vector DA points in the opposite direction.

$$\overrightarrow{AC} = \frac{3}{6}(-\mathbf{n}) = -\frac{1}{2}\mathbf{n}$$

AC has three of the six equal parts in AD .

(b) $\overrightarrow{BD} = \frac{4}{6}(-\mathbf{n}) = -\frac{2}{3}\mathbf{n}$

BD has four of the six parts.

(c) $\overrightarrow{CB} = \frac{1}{6}\mathbf{n}$

CB has one part and points in the direction of DA .

Q3

(a) $\overrightarrow{AB} = \frac{1}{3}(6\mathbf{m}) = 2\mathbf{m}$

AB has one of the three ratio parts in AD .

$$\overrightarrow{BD} = \frac{2}{3}(6\mathbf{m}) = 4\mathbf{m}$$

BD has the other two parts.

$$\overrightarrow{BC} = \frac{1}{2}(4\mathbf{m}) = 2\mathbf{m}$$

C is the midpoint of BD .

$$\overrightarrow{AC} = 2\mathbf{m} + 2\mathbf{m} = 4\mathbf{m}$$

Follow A to B to C .

(b) $\overrightarrow{CA} = -4\mathbf{m}$

Reverse AC .

(c) $\overrightarrow{BC} = 2\mathbf{m}$

Use the midpoint result.

Q4

$$(a) \overrightarrow{BC} = \frac{3}{2}(4\mathbf{m}) = 6\mathbf{m}$$

$$\overrightarrow{AB} = \frac{1}{3}(6\mathbf{m}) = 2\mathbf{m}$$

$$\overrightarrow{AD} = 2\mathbf{m} + 6\mathbf{m} + 4\mathbf{m} = 12\mathbf{m}$$

$$(b) \overrightarrow{DA} = -12\mathbf{m}$$

$$(c) \overrightarrow{AC} = 2\mathbf{m} + 6\mathbf{m} = 8\mathbf{m}$$

CD represents two ratio parts, so each part is $2\mathbf{m}$.

AB points in the same direction as BC .

Follow A to B to C to D .

Reverse AD .

Follow A to B to C .

Q5

$$(a) \overrightarrow{AB} = -\mathbf{m} + \mathbf{n}$$

$$\overrightarrow{AP} = \frac{1}{4}(-\mathbf{m} + \mathbf{n})$$

$$\overrightarrow{OP} = \mathbf{m} + \frac{1}{4}(-\mathbf{m} + \mathbf{n})$$

$$\overrightarrow{OP} = \frac{3}{4}\mathbf{m} + \frac{1}{4}\mathbf{n}$$

$$(b) \overrightarrow{PB} = \frac{3}{4}(-\mathbf{m} + \mathbf{n})$$

$$\overrightarrow{PB} = -\frac{3}{4}\mathbf{m} + \frac{3}{4}\mathbf{n}$$

Follow A to O to B .

AP has one of the four equal parts of AB .

Follow O to A to P .

Expand and collect terms.

PB has three of the four equal parts.

Multiply both terms by three quarters.

Q6

$$(a) \overrightarrow{OP} = \frac{3}{5}\mathbf{m}$$

$$\overrightarrow{OQ} = \frac{2}{3}\mathbf{n}$$

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$$

$$\overrightarrow{PQ} = -\frac{3}{5}\mathbf{m} + \frac{2}{3}\mathbf{n}$$

$$(b) \overrightarrow{QP} = \frac{3}{5}\mathbf{m} - \frac{2}{3}\mathbf{n}$$

OP has three of the five equal parts of OA .

OQ has two of the three equal parts of OB .

Follow P to O to Q .

Reverse OP before adding OQ .

Reverse PQ by changing both signs.

Q7

$$\overrightarrow{AB} = -2\mathbf{m} + 3\mathbf{n}$$

$$\overrightarrow{MA} = \mathbf{m}$$

$$\overrightarrow{AP} = \frac{3}{4}(-2\mathbf{m} + 3\mathbf{n})$$

$$\overrightarrow{MP} = \mathbf{m} + \frac{3}{4}(-2\mathbf{m} + 3\mathbf{n})$$

$$\overrightarrow{MP} = \mathbf{m} - \frac{3}{2}\mathbf{m} + \frac{9}{4}\mathbf{n}$$

$$\overrightarrow{MP} = -\frac{1}{2}\mathbf{m} + \frac{9}{4}\mathbf{n}$$

Follow A to O to B.

M is the midpoint of OA, whose vector is $2\mathbf{m}$.

AP has three of the four equal parts of AB.

Follow M to A to P.

Expand the brackets.

Collect the multiples of $\mathbf{m}$.

Q8

$$(a) \overrightarrow{AD} = \mathbf{m} + \frac{2}{3}(-\mathbf{m} + \mathbf{n})$$

$$\overrightarrow{AD} = \frac{1}{3}\mathbf{m} + \frac{2}{3}\mathbf{n}$$

$$\overrightarrow{DE} = -\overrightarrow{AD} + \overrightarrow{AE}$$

$$\overrightarrow{DE} = -\frac{1}{3}\mathbf{m} - \frac{2}{3}\mathbf{n} + \frac{1}{4}\mathbf{n}$$

$$\overrightarrow{DE} = -\frac{1}{3}\mathbf{m} - \frac{5}{12}\mathbf{n}$$

$$(b) \overrightarrow{DC} = \frac{1}{3}(-\mathbf{m} + \mathbf{n})$$

$$\overrightarrow{CE} = -\frac{3}{4}\mathbf{n}$$

$$\overrightarrow{DE} = -\frac{1}{3}\mathbf{m} + \frac{1}{3}\mathbf{n} - \frac{3}{4}\mathbf{n}$$

Follow A to B to D.

Expand and collect terms.

Follow D to A to E.

Reverse AD and add AE.

Collect the multiples of $\mathbf{n}$.

For a second route, first travel D to C.

CE points backwards along AC.

Add DC and CE. This gives the same result.

Q9

$$\overrightarrow{OC} = -\mathbf{m} + \mathbf{n}$$

$$\overrightarrow{OP} = \frac{2}{3}(-\mathbf{m} + \mathbf{n})$$

$$\overrightarrow{OQ} = \mathbf{m} + \frac{1}{3}(-\mathbf{m} + \mathbf{n})$$

$$\overrightarrow{OQ} = \frac{2}{3}\mathbf{m} + \frac{1}{3}\mathbf{n}$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$\overrightarrow{PQ} = \frac{4}{3}\mathbf{m} - \frac{1}{3}\mathbf{n}$$

OB is the diagonal; subtract OA to get OC.

OP has two thirds of OC.

Follow O to A to Q.

Expand and collect terms.

Follow P to O to Q.

Subtract both coefficients of OP.

Q10

(a) $\overrightarrow{BC} = -\mathbf{m} + \mathbf{n} + \frac{3}{2}\mathbf{m}$

$$\overrightarrow{BC} = \frac{1}{2}\mathbf{m} + \mathbf{n}$$

$$\overrightarrow{AN} = \mathbf{m} + \frac{1}{3}\left(\frac{1}{2}\mathbf{m} + \mathbf{n}\right)$$

$$\overrightarrow{AN} = \frac{7}{6}\mathbf{m} + \frac{1}{3}\mathbf{n}$$

$$\overrightarrow{MN} = \overrightarrow{AN} - \overrightarrow{AM}$$

$$\overrightarrow{MN} = \frac{7}{6}\mathbf{m} - \frac{1}{6}\mathbf{n}$$

(b) $\overrightarrow{NC} = \frac{2}{3}\left(\frac{1}{2}\mathbf{m} + \mathbf{n}\right)$

$$\overrightarrow{NC} = \frac{1}{3}\mathbf{m} + \frac{2}{3}\mathbf{n}$$

Follow B to A to D to C .

Collect the multiples of $\mathbf{m}$.

Follow A to B to N ; BN is one third of BC .

Expand and collect terms.

Follow M to A to N .

Subtract half of $\mathbf{n}$, because M is the midpoint of AD .

NC has two thirds of BC .

Multiply both terms by two thirds.

Q11

(a) $\overrightarrow{AD} = \frac{2}{5}\mathbf{m}$

$$\overrightarrow{AE} = \mathbf{n} + \frac{1}{4}\mathbf{n} = \frac{5}{4}\mathbf{n}$$

$$\overrightarrow{DE} = -\frac{2}{5}\mathbf{m} + \frac{5}{4}\mathbf{n}$$

$$\overrightarrow{DF} = -\frac{1}{5}\mathbf{m} + \frac{5}{8}\mathbf{n}$$

$$\overrightarrow{AF} = \frac{2}{5}\mathbf{m} - \frac{1}{5}\mathbf{m} + \frac{5}{8}\mathbf{n}$$

$$\overrightarrow{AF} = \frac{1}{5}\mathbf{m} + \frac{5}{8}\mathbf{n}$$

$$\overrightarrow{BF} = -\frac{4}{5}\mathbf{m} + \frac{5}{8}\mathbf{n}$$

(b) $\overrightarrow{CF} = \frac{1}{5}\mathbf{m} - \frac{3}{8}\mathbf{n}$

AD has two of the five equal parts of AB .

Add AC and the extension CE .

Follow D to A to E .

F is the midpoint, so halve DE .

Follow A to D to F .

Collect the multiples of $\mathbf{m}$.

Follow B to A to F : subtract $\mathbf{m}$ from AF .

Follow C to A to F : subtract $\mathbf{n}$ from AF .

Q12

(a) $\overrightarrow{OM} = \mathbf{m} + \frac{2}{3}\mathbf{n}$

$$\overrightarrow{ON} = \frac{1}{3}\mathbf{n}$$

$$\overrightarrow{MN} = -\mathbf{m} - \frac{1}{3}\mathbf{n}$$

$$\overrightarrow{MT} = -\frac{1}{2}\mathbf{m} - \frac{1}{6}\mathbf{n}$$

$$\overrightarrow{OT} = \mathbf{m} + \frac{2}{3}\mathbf{n} - \frac{1}{2}\mathbf{m} - \frac{1}{6}\mathbf{n}$$

Follow O to A to M .

ON has one third of OC .

Follow M to O to N .

T is the midpoint, so halve MN .

Follow O to M to T .

Q12 continued

$$\overrightarrow{OT} = \frac{1}{2}\mathbf{m} + \frac{1}{2}\mathbf{n}$$

(b) $\overrightarrow{OB} = \mathbf{m} + \mathbf{n}$

$$\overrightarrow{OT} = \frac{1}{2}\overrightarrow{OB}$$

Collect the multiples of each vector.

Find the whole diagonal.

Starting from the same point O , T is halfway to B .
It is the midpoint.

Q13

$$x - 2 = 4$$

Compare the coefficients of m .

$$x = 6$$

Add 2 to both sides.

$$y + 3 = -2$$

Compare the coefficients of n .

$$y = -5$$

Subtract 3 from both sides.

Q14

$$2p + 1 = 7$$

Compare the coefficients of m .

$$2p = 6, \quad p = 3$$

Subtract 1, then divide by 2.

$$3 - q = 0$$

The right-hand side has no n term, so its
coefficient is zero.

$$q = 3$$

Add q to both sides.**Q15**

$$x(\mathbf{m} + \mathbf{n}) + 3(\mathbf{m} - \mathbf{n}) = 7\mathbf{m} + \mathbf{n}$$

Substitute $y = 3$.

$$xm + xn + 3m - 3n = 7m + n$$

Expand both brackets.

$$(x + 3)\mathbf{m} + (x - 3)\mathbf{n} = 7\mathbf{m} + \mathbf{n}$$

Collect the multiples of each vector.

$$x + 3 = 7$$

Compare the coefficients of m .

$$x = 4$$

Subtract 3.

$$4 - 3 = 1$$

The coefficient of n also matches.

Q16

(a) $xm + y(2m) = 6m$

$$(x + 2y)m = 6m$$

$$x + 2y = 6$$

$$x + 2 = 6, \quad x = 4$$

(b) $x + 4 = 6, \quad x = 2$

Replace n by $2m$, as this question instructs.

Collect the multiples of m .

The coefficient of m must be 6.

Substitute $y = 1$.

Substitute $y = 2$.

Q17

$$(3x + y)m + (x - 2y)n = 11m - n$$

$$3x + y = 11$$

$$x - 2y = -1$$

$$x = 2y - 1$$

$$3(2y - 1) + y = 11$$

$$7y = 14, \quad y = 2$$

$$x = 3$$

Expand the brackets and collect terms.

Compare the coefficients of m .

Compare the coefficients of n .

Rearrange the second equation.

Substitute into the first equation.

Expand and solve.

Substitute $y = 2$ into $x = 2y - 1$.

Q18

$$x - 2y = -5$$

$$3x + y = 13$$

$$x = 2y - 5$$

$$3(2y - 5) + y = 13$$

$$7y = 28, \quad y = 4$$

$$x = 3$$

Compare the coefficients of m .

Compare the coefficients of n .

Rearrange the first equation.

Substitute into the second equation.

Expand and solve.

Substitute $y = 4$ into $x = 2y - 5$.

Q19

$$\left(\frac{x}{2} - 2y\right)\mathbf{m} + (x + y)\mathbf{n} = -\mathbf{m} + 8\mathbf{n}$$

$$\frac{x}{2} - 2y = -1$$

$$x + y = 8$$

$$x - 4y = -2$$

$$5y = 10, \quad y = 2$$

$$x = 6$$

Expand both brackets, including the negative multiplier.

Compare the coefficients of $\mathbf{m}$.

Compare the coefficients of $\mathbf{n}$.

Multiply the first equation by 2.

Subtract this equation from $x + y = 8$ to eliminate x .

Substitute $y = 2$ into $x + y = 8$.

Q20

$$(4x + y)\mathbf{m} + (-x + 2y)\mathbf{n} = 10\mathbf{m} - 7\mathbf{n}$$

$$4x + y = 10$$

$$-x + 2y = -7$$

$$y = 10 - 4x$$

$$-x + 2(10 - 4x) = -7$$

$$-9x = -27, \quad x = 3$$

$$y = -2$$

Expand and collect all the multiples of each vector.

Compare the coefficients of $\mathbf{m}$.

Compare the coefficients of $\mathbf{n}$.

Rearrange the first equation.

Substitute into the second equation.

Expand and solve.

Substitute $x = 3$ into $y = 10 - 4x$.

Q21

(a) $\mathbf{u} = \lambda\mathbf{v}$

$$p - 1 = 3\lambda$$

$$8 = 2\lambda$$

$$\lambda = 4$$

$$p = 13$$

(b) $\mathbf{u} = 4\mathbf{v}$

Introduce the unknown scale factor.

Compare the coefficients of $\mathbf{m}$.

Compare the coefficients of $\mathbf{n}$.

Divide the second equation by 2.

Substitute $\lambda = 4$ into $p - 1 = 3\lambda$.

Four times as long, in the same direction.

Q22

(a) $u = \lambda v$

$$4 = -2\lambda$$

$$q + 1 = 3\lambda$$

$$\lambda = -2$$

$$q = -7$$

(b) $u = -2v$

Introduce the unknown scale factor.

Compare the coefficients of m .

Compare the coefficients of n .

Divide the first equation by -2.

Substitute $\lambda = -2$ into $q + 1 = 3\lambda$.

Twice as long, in the opposite direction.

Q23

(a) $u = \lambda v$

$$k + 2 = 2\lambda$$

$$2k - 1 = 5\lambda$$

$$5(k + 2) = 2(2k - 1)$$

$$5k + 10 = 4k - 2$$

$$k = -12$$

(b) $u = -5v$

Introduce the unknown scale factor.

Compare the coefficients of m .

Compare the coefficients of n .

Both sides equal 10λ , so eliminate λ .

Expand both brackets.

Collect terms and solve.

Five times as long, in the opposite direction.

Q24

(a) $u = \lambda v$

$$2t - 1 = 3\lambda$$

$$t + 2 = -\lambda$$

$$2t - 1 = -3(t + 2)$$

$$5t = -5, \quad t = -1$$

(b) $u = -v$

(c) $\lambda = -1$

Introduce the unknown scale factor.

Compare the coefficients of m .

Compare the coefficients of n .

Eliminate λ using the second equation.

Expand and solve.

The same length, in the opposite direction.

From part (b), $u = -v$.

Q25

$$\overrightarrow{AP} = t\overrightarrow{AB}$$

$$\overrightarrow{AB} = -\mathbf{m} + \mathbf{n}$$

$$\overrightarrow{OP} = \mathbf{m} + t(-\mathbf{m} + \mathbf{n})$$

$$\overrightarrow{OM} = \frac{3}{5}\mathbf{m}$$

$$\overrightarrow{MP} = \left(\frac{2}{5} - t\right)\mathbf{m} + t\mathbf{n}$$

$$\overrightarrow{MP} = \lambda\mathbf{n}$$

$$\frac{2}{5} - t = 0$$

$$t = \frac{2}{5}$$

$$AP : PB = 2 : 3$$

Represent the unknown fraction of AB .

Follow A to O to B .

Follow O to A to P .

Use the given ratio $OM : MA = 3 : 2$.

Subtract OM from OP and collect terms.

MP is parallel to OB , whose vector is $\mathbf{n}$.

Compare the coefficients of $\mathbf{m}$.

Solve for the fraction AP/AB .

The remaining fraction PB/AB is three fifths.

Q26

$$\overrightarrow{AB} = -2\mathbf{m} + 3\mathbf{n}$$

$$\overrightarrow{OP} = 2\mathbf{m} + t(-2\mathbf{m} + 3\mathbf{n})$$

$$\overrightarrow{OP} = (2 - 2t)\mathbf{m} + 3t\mathbf{n}$$

$$2 - 2t = 1 + s$$

$$3t = 2s + 1$$

$$s = 1 - 2t$$

$$3t = 2(1 - 2t) + 1$$

$$7t = 3, \quad t = \frac{3}{7}$$

$$s = \frac{1}{7}$$

Follow A to O to B .

Follow O to A to P .

Expand and collect terms.

Compare the $\mathbf{m}$ coefficient with the supplied expression for OP .

Compare the $\mathbf{n}$ coefficient. We have two simultaneous equations.

Rearrange the first equation.

Substitute into the second equation.

Expand and solve.

Substitute t into $s = 1 - 2t$.

Q27

$$\overrightarrow{AC} = -\mathbf{m} + \mathbf{n}$$

$$\overrightarrow{AX} = t\overrightarrow{AC}$$

$$\overrightarrow{OX} = (1-t)\mathbf{m} + t\mathbf{n}$$

$$\overrightarrow{OM} = \frac{1}{2}\mathbf{m} + \mathbf{n}$$

$$\overrightarrow{MX} = \left(\frac{1}{2} - t\right)\mathbf{m} + (t-1)\mathbf{n}$$

$$\overrightarrow{OB} = \mathbf{m} + \mathbf{n}$$

$$\frac{1}{2} - t = \lambda, \quad t - 1 = \lambda$$

$$\frac{1}{2} - t = t - 1$$

$$t = \frac{3}{4}$$

$$AX : XC = 3 : 1$$

Follow A to O to C.

Represent the unknown fraction of AC.

Follow O to A to X and simplify.

M is halfway along CB.

Subtract OM from OX.

Find the diagonal to which MX is parallel.

Write $\overrightarrow{MX} = \lambda \overrightarrow{OB}$ and compare coefficients.

Both coefficients equal the same multiplier.

Solve the equation.

AX is three quarters of AC; XC is one quarter.

Q28

$$(a) \overrightarrow{AD} = \mathbf{m} + \frac{2}{5}(-\mathbf{m} + \mathbf{n})$$

$$\overrightarrow{AD} = \frac{3}{5}\mathbf{m} + \frac{2}{5}\mathbf{n}$$

$$\overrightarrow{AE} = t\mathbf{n}$$

$$\overrightarrow{DE} = -\frac{3}{5}\mathbf{m} + \left(t - \frac{2}{5}\right)\mathbf{n}$$

$$2k = -\frac{3}{5}$$

$$k = -\frac{3}{10}$$

$$(b) -k = t - \frac{2}{5}$$

$$t = \frac{7}{10}$$

$$AE : EC = 7 : 3$$

Follow A to B to D.

Expand and collect terms.

Let t be the fraction of AC from A to E.

Follow D to A to E.

Compare the $\mathbf{m}$ coefficient with $\overrightarrow{DE} = k(2\mathbf{m} - \mathbf{n})$.

Divide by 2.

Compare the coefficients of $\mathbf{n}$.

Substitute k and solve.

AE is seven tenths of AC; EC is three tenths.

Q29

(a) $\overrightarrow{AQ} = -\mathbf{m} + \frac{2}{3}\mathbf{n}$

$$\overrightarrow{BP} = \frac{1}{3}\mathbf{m} - \mathbf{n}$$

$$\overrightarrow{AR} = t\overrightarrow{AQ}, \quad \overrightarrow{BR} = s\overrightarrow{BP}$$

$$\overrightarrow{OR} = (1-t)\mathbf{m} + \frac{2t}{3}\mathbf{n}$$

$$\overrightarrow{OR} = \frac{s}{3}\mathbf{m} + (1-s)\mathbf{n}$$

$$1-t = \frac{s}{3}, \quad \frac{2t}{3} = 1-s$$

$$s = 3 - 3t$$

$$\frac{2t}{3} = 3t - 2$$

$$t = \frac{6}{7}, \quad s = \frac{3}{7}$$

$$AR : RQ = 6 : 1$$

(b) $BR : RP = 3 : 4$

Follow A to O to Q.

Follow B to O to P.

Use a separate unknown for each line through R.

Follow O to A to R.

Follow O to B to R.

Compare coefficients to get simultaneous equations.

Rearrange the first equation.

Substitute s into the second equation.

Multiply by 3 and solve, then find s .

AR is six sevenths of AQ .

BR is three sevenths of BP .

Q30

(a) $\overrightarrow{BC} = \frac{1}{2}\mathbf{m} + \mathbf{n}$

$$\overrightarrow{AF} = \mathbf{m} + \frac{2}{3}\left(\frac{1}{2}\mathbf{m} + \mathbf{n}\right)$$

$$\overrightarrow{AF} = \frac{4}{3}\mathbf{m} + \frac{2}{3}\mathbf{n}$$

$$\overrightarrow{BE} = -\mathbf{m} + \frac{1}{3}\mathbf{n}$$

$$\overrightarrow{AP} = t\overrightarrow{AF}, \quad \overrightarrow{BP} = s\overrightarrow{BE}$$

$$\overrightarrow{AP} = \frac{4t}{3}\mathbf{m} + \frac{2t}{3}\mathbf{n}$$

$$\overrightarrow{AP} = (1-s)\mathbf{m} + \frac{s}{3}\mathbf{n}$$

$$\frac{4t}{3} = 1-s, \quad \frac{2t}{3} = \frac{s}{3}$$

$$s = 2t$$

$$\frac{10t}{3} = 1$$

Follow B to A to D to C.

Follow A to B to F; BF is two thirds of BC .

Expand and collect terms.

Follow B to A to E.

Describe P on each of its two lines.

Use the line AF .

Use the route A to B to P.

Compare coefficients to get simultaneous equations.

Multiply the second equation by 3.

Substitute $s = 2t$ into the first equation and collect terms.

Q30 continued

$$t = \frac{3}{10}, \quad s = \frac{3}{5}$$

$$AP : PF = 3 : 7$$

(b) $BP : PE = 3 : 2$

Solve for t , then find s . AP is three tenths of AF . BP is three fifths of BE .

Q31

(a) $\overrightarrow{AD} = \mathbf{m} + \frac{1}{3}(-\mathbf{m} + \mathbf{n})$

$$\overrightarrow{AD} = \frac{2}{3}\mathbf{m} + \frac{1}{3}\mathbf{n}$$

$$\overrightarrow{BE} = -\mathbf{m} + \frac{2}{5}\mathbf{n}$$

$$\overrightarrow{AP} = t\overrightarrow{AD}, \quad \overrightarrow{BP} = s\overrightarrow{BE}$$

$$\overrightarrow{AP} = \frac{2t}{3}\mathbf{m} + \frac{t}{3}\mathbf{n}$$

$$\overrightarrow{AP} = (1-s)\mathbf{m} + \frac{2s}{5}\mathbf{n}$$

$$\frac{2t}{3} = 1-s, \quad \frac{t}{3} = \frac{2s}{5}$$

$$t = \frac{6s}{5}$$

$$\frac{4s}{5} = 1-s$$

$$s = \frac{5}{9}, \quad t = \frac{2}{3}$$

$$\overrightarrow{AP} = \frac{4}{9}\mathbf{m} + \frac{2}{9}\mathbf{n}$$

(b) $BP : PE = 5 : 4$

Follow A to B to D.

Expand and collect terms.

Follow B to A to E.

Describe P on the two lines.

Use the line AD.

Use the route A to B to P.

Compare coefficients.

Rearrange the second equation.

Substitute into the first equation.

Solve for s , then find t .Substitute t into the first expression for AP . BP is five ninths of BE .

Q32

(a) $\overrightarrow{CD} = \frac{2}{3}\mathbf{m} - \mathbf{n}$

$$\overrightarrow{BE} = -\mathbf{m} + 2\mathbf{n}$$

$$\overrightarrow{CP} = t\overrightarrow{CD}, \quad \overrightarrow{BP} = s\overrightarrow{BE}$$

$$\overrightarrow{AP} = \frac{2t}{3}\mathbf{m} + (1-t)\mathbf{n}$$

$$\overrightarrow{AP} = (1-s)\mathbf{m} + 2s\mathbf{n}$$

$$\frac{2t}{3} = 1-s, \quad 1-t = 2s$$

Follow C to A to D.

 AE is twice AC . Follow B to A to E.

The multipliers may be negative or greater than 1 on extended lines.

Follow A to C to P.

Follow A to B to P.

Compare coefficients to get simultaneous equations.

Q32 continued

$$s = \frac{1-t}{2}$$

$$4t = 3 + 3t$$

$$t = 3, \quad s = -1$$

$$\overrightarrow{AP} = 2\mathbf{m} - 2\mathbf{n}$$

(b) $\overrightarrow{BP} = -\overrightarrow{BE}$

$$BP : PE = 1 : 2$$

(c) $\overrightarrow{BP} = -\overrightarrow{BE}$

Rearrange the second equation.

Substitute into the first equation and multiply by 6.

Solve for t , then find s .

Substitute into either expression for AP .

P is beyond B , away from E . BP and BE have equal lengths.

PE consists of PB and BE , so is twice BP .

P lies beyond B , on the opposite side of B from E . The minus sign shows that BP and BE point in opposite directions.

Q33

$$\overrightarrow{AF} = \frac{3}{4}\mathbf{m} + \mathbf{n}$$

$$\overrightarrow{DE} = \frac{2}{5}\mathbf{m} - \mathbf{n}$$

$$\overrightarrow{AP} = t\overrightarrow{AF}, \quad \overrightarrow{DP} = s\overrightarrow{DE}$$

$$\frac{3t}{4} = \frac{2s}{5}, \quad t = 1 - s$$

$$s = 1 - t$$

$$15t = 8 - 8t$$

$$t = \frac{8}{23}$$

$$\overrightarrow{AP} = \frac{6}{23}\mathbf{m} + \frac{8}{23}\mathbf{n}$$

$$\overrightarrow{BP} = -\frac{17}{23}\mathbf{m} + \frac{8}{23}\mathbf{n}$$

$$\overrightarrow{BQ} = u\overrightarrow{BP}$$

$$\overrightarrow{AQ} = \left(1 - \frac{17u}{23}\right)\mathbf{m} + \frac{8u}{23}\mathbf{n}$$

$$1 - \frac{17u}{23} = 0$$

Follow A to D to F .

Follow D to A to E .

Describe P on the two lines.

Compare coefficients in the two expressions for AP .

Rearrange the second equation.

Substitute for s in the first equation and multiply by 20.

Solve $23t = 8$.

Substitute t into $\overrightarrow{AP} = t\overrightarrow{AF}$.

Subtract AB from AP .

Introduce a new multiplier to locate Q .

Follow A to B to Q and collect terms.

Q lies on AD , so AQ has no $\mathbf{m}$ term.

Q33 continued

$$u = \frac{23}{17}$$

$$\overrightarrow{AQ} = \frac{8}{17}\mathbf{n}$$

$$AQ : QD = 8 : 9$$

Solve for u .

Substitute into the remaining coefficient.

QD is the remaining nine seventeenths of AD .

Q34

$$\overrightarrow{AE} = \frac{3}{2}\mathbf{m} + \frac{1}{2}\mathbf{n}$$

$$\overrightarrow{BF} = -\mathbf{m} + \frac{1}{3}\mathbf{n}$$

$$\overrightarrow{AP} = t\overrightarrow{AE}, \quad \overrightarrow{BP} = s\overrightarrow{BF}$$

$$\frac{3t}{2} = 1 - s, \quad \frac{t}{2} = \frac{s}{3}$$

$$s = \frac{3t}{2}$$

$$3t = 1, \quad t = \frac{1}{3}$$

$$\overrightarrow{AP} = \frac{1}{2}\mathbf{m} + \frac{1}{6}\mathbf{n}$$

$$\overrightarrow{AC} = 2\mathbf{m} + \mathbf{n}$$

$$\overrightarrow{CP} = -\frac{3}{2}\mathbf{m} - \frac{5}{6}\mathbf{n}$$

$$\overrightarrow{CQ} = u\overrightarrow{CP}$$

$$\overrightarrow{AQ} = (2 - \frac{3u}{2})\mathbf{m} + (1 - \frac{5u}{6})\mathbf{n}$$

$$1 - \frac{5u}{6} = 0$$

$$u = \frac{6}{5}$$

$$\overrightarrow{AQ} = \frac{1}{5}\mathbf{m}$$

$$AQ : QB = 1 : 4$$

Follow A to B to E , using E as the midpoint of BC .

Follow B to A to F .

Describe P on the two lines.

Compare coefficients in the two expressions for AP .

Rearrange the second equation.

Substitute into the first equation and solve.

Substitute t into $\overrightarrow{AP} = t\overrightarrow{AE}$.

Follow A to D to C .

Subtract AC from AP .

Introduce a new multiplier to locate Q .

Follow A to C to Q and collect terms.

Q lies on AB , so AQ has no $\mathbf{n}$ term.

Solve for u .

Substitute into the remaining coefficient.

QB is the remaining four fifths of AB .

Q35

$$(a) \quad \overrightarrow{OM} = \frac{1}{3}\mathbf{m} + \frac{2}{3}\mathbf{n}$$

$$\overrightarrow{OP} = \frac{1}{4}\mathbf{m} + \frac{1}{2}\mathbf{n}$$

$$\overrightarrow{AP} = -\frac{3}{4}\mathbf{m} + \frac{1}{2}\mathbf{n}$$

$$\overrightarrow{AQ} = t\overrightarrow{AP}$$

$$\overrightarrow{OQ} = (1 - \frac{3t}{4})\mathbf{m} + \frac{t}{2}\mathbf{n}$$

$$1 - \frac{3t}{4} = 0, \quad t = \frac{4}{3}$$

$$\overrightarrow{OQ} = \frac{2}{3}\mathbf{n}$$

$$OQ : QB = 2 : 1$$

$$(b) \quad \overrightarrow{BP} = \frac{1}{4}\mathbf{m} - \frac{1}{2}\mathbf{n}$$

$$\overrightarrow{BR} = s\overrightarrow{BP}$$

$$\overrightarrow{OR} = \frac{s}{4}\mathbf{m} + (1 - \frac{s}{2})\mathbf{n}$$

$$1 - \frac{s}{2} = 0, \quad s = 2$$

$$\overrightarrow{OR} = \frac{1}{2}\mathbf{m}$$

$$OR : RA = 1 : 1$$

$$(c) \quad \overrightarrow{QR} = \frac{1}{2}\mathbf{m} - \frac{2}{3}\mathbf{n}$$

$$\overrightarrow{AB} = -\mathbf{m} + \mathbf{n}$$

$$\lambda = -\frac{1}{2}, \quad \lambda = -\frac{2}{3}$$

Follow O to A to M .

OP is three quarters of OM .

Subtract OA from OP .

Use the line AP to locate Q .

Follow O to A to Q .

Q lies on OB , so the $\mathbf{m}$ coefficient is zero.

Substitute t into the remaining coefficient.

OQ is two thirds of OB .

Subtract OB from OP .

Use the line BP to locate R .

Follow O to B to R .

R lies on OA , so the $\mathbf{n}$ coefficient is zero.

Substitute s into the remaining coefficient.

R is halfway along OA .

Subtract OQ from OR .

Compare QR with the whole side AB .

These would both be needed for $\overrightarrow{QR} = \lambda \overrightarrow{AB}$. One number cannot have both values, so QR is not parallel to AB .

Q36

$$(a) \quad \overrightarrow{AQ} = \frac{5}{11}\mathbf{n}$$

$$\overrightarrow{BQ} = -\mathbf{m} + \frac{5}{11}\mathbf{n}$$

$$\overrightarrow{BP} = u\overrightarrow{BQ}$$

$$\overrightarrow{AP} = (1 - u)\mathbf{m} + \frac{5u}{11}\mathbf{n}$$

$$\overrightarrow{DP} = s\overrightarrow{DE}$$

The ratio 5:6 gives AQ five of the eleven parts of AD .

Follow B to A to Q .

Introduce a multiplier to locate P on BQ .

Follow A to B to P .

P also lies on DE .

Q36 continued

$$\overrightarrow{AP} = \frac{s}{3}\mathbf{m} + (1-s)\mathbf{n}$$

$$1-u = \frac{s}{3}, \quad \frac{5u}{11} = 1-s$$

$$s = 3 - 3u$$

$$5u = 33u - 22$$

$$u = \frac{11}{14}, \quad s = \frac{9}{14}$$

$$\overrightarrow{AP} = \frac{3}{14}\mathbf{m} + \frac{5}{14}\mathbf{n}$$

$$\overrightarrow{AF} = \frac{14}{5}\overrightarrow{AP}$$

$$\overrightarrow{AF} = \frac{3}{5}\mathbf{m} + \mathbf{n}$$

$$DF : FC = 3 : 2, \quad r = \frac{3}{2}$$

(b) $f = \frac{r}{r+1}$

$$\overrightarrow{AF} = f\mathbf{m} + \mathbf{n}$$

$$\overrightarrow{AP} = t\overrightarrow{AF}$$

$$\overrightarrow{DP} = s\overrightarrow{DE}$$

$$tf = \frac{s}{3}, \quad t = 1-s$$

$$s = 3tf$$

$$t(1+3f) = 1$$

$$t = \frac{1}{1+3f}$$

$$\overrightarrow{AP} = \frac{f}{1+3f}\mathbf{m} + \frac{1}{1+3f}\mathbf{n}$$

$$\overrightarrow{BP} = -\frac{1+2f}{1+3f}\mathbf{m} + \frac{1}{1+3f}\mathbf{n}$$

$$\overrightarrow{BQ} = k\overrightarrow{BP}$$

$$1 - \frac{k(1+2f)}{1+3f} = 0$$

$$k = \frac{1+3f}{1+2f}$$

Follow A to D to P, using $\overrightarrow{DE} = \frac{1}{3}\mathbf{m} - \mathbf{n}$.

Compare coefficients to get simultaneous equations.

Rearrange the first equation.

Substitute into the second equation and multiply by 11.

Solve for u , then find s .

Substitute into either expression for AP .

F lies on DC , so AF has $\mathbf{n}$ coefficient 1. Multiply AP by $14/5$.

Multiply both coefficients.

DF is three fifths of DC . To write 3:2 as $r:1$, divide both parts by 2.

Let f be the fraction of DC from D to F .

Follow A to D to F.

Use t for the fraction of AF from A to P.

Use s for the fraction of DE from D to P.

Compare coefficients in the two expressions for AP .

Rearrange the first equation.

Substitute $s = 3tf$ into the second equation and collect terms.

Divide by $1+3f$, which is positive.

Substitute t into $\overrightarrow{AP} = t\overrightarrow{AF}$.

Subtract AB from AP .

Use a new multiplier k to locate Q .

Q lies on AD . In $\overrightarrow{AQ} = \overrightarrow{AB} + \overrightarrow{BQ}$, the $\mathbf{m}$ coefficient must be zero.

Rearrange to find k .

Q36 continued

$$\overrightarrow{AQ} = \frac{1}{1+2f} \mathbf{n}$$

$$\frac{AQ}{AD} = \frac{r+1}{3r+1}$$

$$\frac{QD}{AD} = \frac{2r}{3r+1}$$

$$AQ : QD = (r+1) : 2r$$

Substitute k into the $\mathbf{n}$ coefficient of AQ .

Replace f by $r/(r+1)$ and simplify.

Subtract AQ/AD from 1 to find the remaining fraction.

The two fractions have the same denominator, so compare their numerators.