

Vector Ratios and Intersections

Long workbook

Problems using geometric diagrams

Questions on GCSE Maths topics, including challenges beyond typical exam questions.

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Vector basics recap

A vector has a length and a direction. Moving an arrow to a different place does not change the vector, provided its length and direction stay the same.

$\overrightarrow{AB}$ is the vector from A to B; AB is the length of the line segment. Bold italic letters, such as $\mathbf{m}$ and $\mathbf{n}$, also stand for vectors.

Multiplying a vector

 $\mathbf{m}$ 

The original vector.

 $2\mathbf{m}$ 

Twice the length, in the same direction.

 $\frac{1}{2}\mathbf{m}$ 

Half the length, in the same direction.

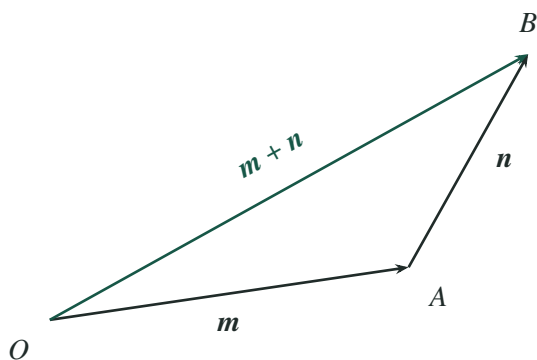
 $-\mathbf{m}$ 

The same length, in the opposite direction.

A positive multiplier keeps the direction; a negative multiplier reverses it.

Adding vectors

Follow the first vector, then the second. The sum takes you directly from the starting point to the finishing point.



$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$

$$\overrightarrow{OB} = \mathbf{m} + \mathbf{n}$$

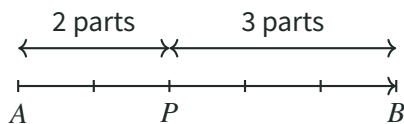
O to A , then A to B , gives the same movement as O to B .

To subtract a vector, add its opposite. For example, $\mathbf{m} - \mathbf{n} = \mathbf{m} + (-\mathbf{n})$.

Vector basics recap

From a ratio to a fraction

If P lies between A and B with $AP : PB = 2 : 3$, the whole AB has five equal parts. Thus $\overrightarrow{AP} = \frac{2}{5}\overrightarrow{AB}$, in the same direction.



Lengths and directions

A length ratio gives sizes. Use the order of the points to find the vector directions. For A, B, C in that order, $BA = \frac{1}{3}BC$ gives $\overrightarrow{BA} = -\frac{1}{3}\overrightarrow{BC}$: BA points back towards A .

Parallel vectors

Two non-zero vectors are parallel when one is a scalar multiple of the other: $\mathbf{u} = \lambda\mathbf{v}$. The same multiplier λ applies to the whole vector. Ignore the sign to find the length factor. A negative sign reverses the direction.

Why coefficient comparison works

Suppose $a\mathbf{m} + b\mathbf{n} = c\mathbf{m} + d\mathbf{n}$. Rearranging gives $(a - c)\mathbf{m} = (d - b)\mathbf{n}$. If $\mathbf{m}$ and $\mathbf{n}$ are non-zero and non-parallel, a non-zero multiple of one cannot equal a multiple of the other. Both coefficients must therefore be zero: $a = c$ and $b = d$.

1A

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Reversing vectors and using ratios

Example Problem

A, B, C and D lie on a straight line in that order.

$AB : BC = 2 : 3$, $CD = \frac{1}{2}AC$ and $\overrightarrow{AC} = \mathbf{m}$.

(a) Express $\overrightarrow{BA}$ in terms of $\mathbf{m}$.

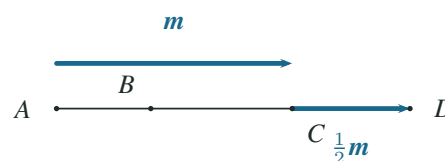
(b) Express $\overrightarrow{BD}$ in terms of $\mathbf{m}$.

(c) Express $\overrightarrow{DA}$ in terms of $\mathbf{m}$.



The idea

1. Use the ratio to find the fraction of the whole segment represented by each part.
2. Build a longer vector by adding vectors along a route.
3. Reverse a vector by changing its sign.



$$AB : BC = 2 : 3$$

Worked Solution

(a) $AB : BC = 2 : 3$

$$\overrightarrow{AB} = \frac{2}{5}\mathbf{m}$$

$$\overrightarrow{BA} = -\frac{2}{5}\mathbf{m}$$

(b) $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$

$$\overrightarrow{BD} = \frac{3}{5}\mathbf{m} + \frac{1}{2}\mathbf{m}$$

$$\overrightarrow{BD} = \frac{11}{10}\mathbf{m}$$

(c) $\overrightarrow{AD} = \mathbf{m} + \frac{1}{2}\mathbf{m} = \frac{3}{2}\mathbf{m}$

$$\overrightarrow{DA} = -\frac{3}{2}\mathbf{m}$$

Of five equal parts in AC , AB has two and BC has three.

AB is two fifths of the whole vector AC .

BA points in the opposite direction to AB .

Travel from B to C , then from C to D .

BC has three of the five parts; CD is half of AC .

Add the fractions using denominator 10.

Travel from A to C , then from C to D .

Reverse the direction to get DA .

1A Reversing vectors and using ratios

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Q1

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A, B, C and D lie on a straight line in that order.

$AB : BC = 3 : 4$, $CD = \frac{2}{7}AC$ and $\overrightarrow{AC} = m$.

(a) Express $\overrightarrow{BA}$ in terms of m .

(b) Express $\overrightarrow{BD}$ in terms of m .

(c) Express $\overrightarrow{DA}$ in terms of m .



Q2

**

A, B, C and D lie on a straight line in that order.

$AB : BC : CD = 2 : 1 : 3$ and $\overrightarrow{DA} = n$.

(a) Express $\overrightarrow{AC}$ in terms of n .

(b) Express $\overrightarrow{BD}$ in terms of n .

(c) Express $\overrightarrow{CB}$ in terms of n .



1A Reversing vectors and using ratios

**

Q3

**

A , B and D lie on a straight line in that order. C is the midpoint of BD .

$\overrightarrow{AD} = 6\mathbf{m}$ and $AB : BD = 1 : 2$.

(a) Express $\overrightarrow{AC}$ in terms of $\mathbf{m}$.

(b) Express $\overrightarrow{CA}$ in terms of $\mathbf{m}$.

(c) Express $\overrightarrow{BC}$ in terms of $\mathbf{m}$.



Q4

**

A , B , C and D lie on a straight line in that order.

$BC : CD = 3 : 2$, $BA = \frac{1}{3}BC$ and $\overrightarrow{CD} = 4\mathbf{m}$.

(a) Express $\overrightarrow{AD}$ in terms of $\mathbf{m}$.

(b) Express $\overrightarrow{DA}$ in terms of $\mathbf{m}$.

(c) Express $\overrightarrow{AC}$ in terms of $\mathbf{m}$.



1B

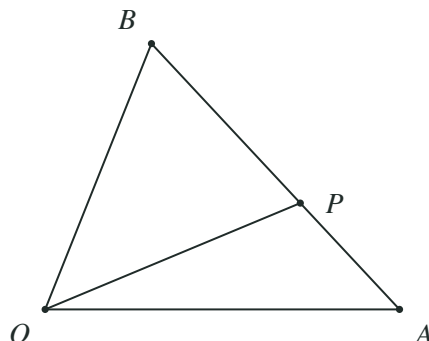
Vector routes in triangles

Example Problem

OAB is a triangle with $\overrightarrow{OA} = \mathbf{m}$ and $\overrightarrow{OB} = \mathbf{n}$.

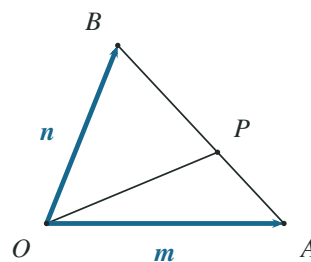
P lies on AB with $AP : PB = 2 : 3$.

Express $\overrightarrow{OP}$ in terms of $\mathbf{m}$ and $\mathbf{n}$.



The idea

1. Find AB using the route A to O to B .
2. Use the given ratio to find AP as a fraction of AB .
3. Add OA and AP to reach P from O .



$$AP : PB = 2 : 3$$

Worked Solution

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$

$$\overrightarrow{AB} = -\mathbf{m} + \mathbf{n}$$

$$\overrightarrow{AP} = \frac{2}{5}\overrightarrow{AB}$$

$$\overrightarrow{AP} = \frac{2}{5}(-\mathbf{m} + \mathbf{n})$$

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

$$\overrightarrow{OP} = \mathbf{m} + \frac{2}{5}(-\mathbf{m} + \mathbf{n})$$

$$\overrightarrow{OP} = \mathbf{m} - \frac{2}{5}\mathbf{m} + \frac{2}{5}\mathbf{n}$$

$$\overrightarrow{OP} = \frac{3}{5}\mathbf{m} + \frac{2}{5}\mathbf{n}$$

Use O to connect the two given vectors.

AO is the reverse of OA .

The ratio 2:3 gives AP two of the five equal parts in AB .

Substitute the expression for AB .

Follow the route O to A to P .

Replace OA and AP by their expressions.

Multiply both terms inside the brackets.

Collect the multiples of each vector.

1B Vector routes in triangles

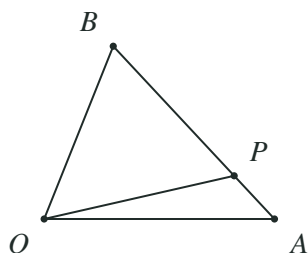
Q5

OAB is a triangle with $\overrightarrow{OA} = \mathbf{m}$ and $\overrightarrow{OB} = \mathbf{n}$.

P lies on AB with $AP : PB = 1 : 3$.

(a) Express $\overrightarrow{OP}$ in terms of $\mathbf{m}$ and $\mathbf{n}$.

(b) Express $\overrightarrow{PB}$ in terms of $\mathbf{m}$ and $\mathbf{n}$.



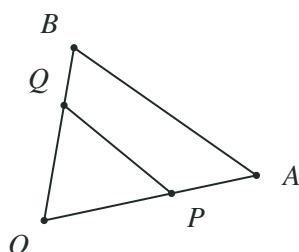
Q6

OAB is a triangle with $\overrightarrow{OA} = \mathbf{m}$ and $\overrightarrow{OB} = \mathbf{n}$.

P lies on OA with $OP : PA = 3 : 2$. Q lies on OB with $OQ : QB = 2 : 1$.

(a) Express $\overrightarrow{PQ}$ in terms of $\mathbf{m}$ and $\mathbf{n}$.

(b) Write down $\overrightarrow{QP}$.



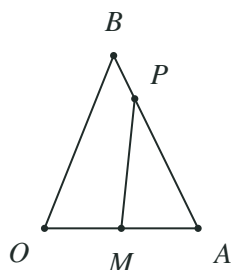
1B Vector routes in triangles

Q7

OAB is a triangle with $\overrightarrow{OA} = 2\mathbf{m}$ and $\overrightarrow{OB} = 3\mathbf{n}$.

M is the midpoint of OA . P lies on AB with $AP : PB = 3 : 1$.

Express $\overrightarrow{MP}$ in terms of $\mathbf{m}$ and $\mathbf{n}$.



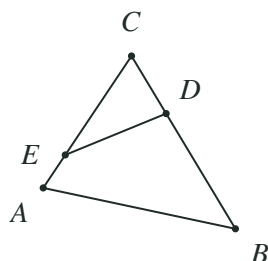
Q8

ABC is a triangle with $\overrightarrow{AB} = \mathbf{m}$ and $\overrightarrow{AC} = \mathbf{n}$.

D lies on BC with $BD : DC = 2 : 1$. E lies on AC with $AE : EC = 1 : 3$.

(a) Express $\overrightarrow{DE}$ in terms of $\mathbf{m}$ and $\mathbf{n}$.

(b) Check your answer using a different route.



1C

Diagonals, midpoints and extensions

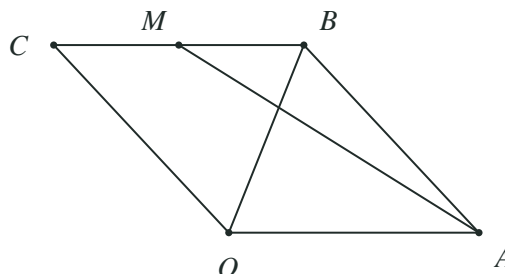
Example Problem

$OABC$ is a parallelogram. $\overrightarrow{OA} = \mathbf{m}$ and $\overrightarrow{OB} = \mathbf{n}$.

M is the midpoint of BC .

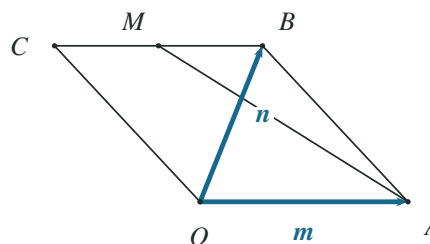
(a) Express $\overrightarrow{AM}$ in terms of $\mathbf{m}$ and $\mathbf{n}$.

(b) Express $\overrightarrow{CM}$ in terms of $\mathbf{m}$ and $\mathbf{n}$.



The idea

1. Notice that the given vector $\mathbf{n}$ is a diagonal, rather than a side.
2. Use a route through O to find AB .
3. Use the midpoint and the directions of opposite sides to reach M .



$$CM : MB = 1 : 1$$

Worked Solution

(a) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$$\overrightarrow{AB} = -\mathbf{m} + \mathbf{n}$$

$$\overrightarrow{BC} = -\mathbf{m}$$

$$\overrightarrow{BM} = -\frac{1}{2}\mathbf{m}$$

$$\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM}$$

$$\overrightarrow{AM} = -\mathbf{m} + \mathbf{n} - \frac{1}{2}\mathbf{m}$$

$$\overrightarrow{AM} = -\frac{3}{2}\mathbf{m} + \mathbf{n}$$

(b) $\overrightarrow{CM} = \frac{1}{2}\mathbf{m}$

Travel from A to O to B . The diagonal OB is $\mathbf{n}$.

Reverse OA , then add OB .

BC has the same length as OA and the opposite direction.

M is the midpoint of BC .

Follow the route A to B to M .

Substitute the two expressions.

Collect the multiples of $\mathbf{m}$.

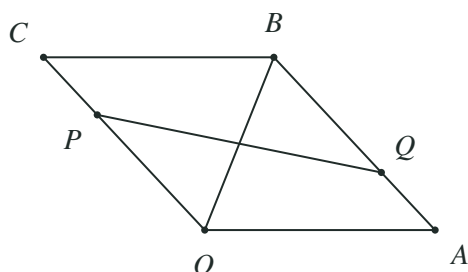
CB has the same length and direction as OA because $OABC$ is a parallelogram. M is the midpoint of CB , so CM is half of OA .

1C Diagonals, midpoints and extensions

Q9

$OABC$ is a parallelogram with $\overrightarrow{OA} = \mathbf{m}$ and $\overrightarrow{OB} = \mathbf{n}$.
 P lies on OC with $OP : PC = 2 : 1$. Q lies on AB
 with $AQ : QB = 1 : 2$.

Express $\overrightarrow{PQ}$ in terms of $\mathbf{m}$ and $\mathbf{n}$.



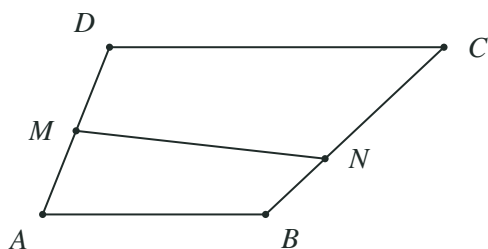
Q10

$ABCD$ is a trapezium with $\overrightarrow{AB} = \mathbf{m}$, $\overrightarrow{AD} = \mathbf{n}$ and
 $\overrightarrow{DC} = \frac{3}{2}\mathbf{m}$.

M is the midpoint of AD . N lies on BC with $BN : NC = 1 : 2$.

(a) Express $\overrightarrow{MN}$ in terms of $\mathbf{m}$ and $\mathbf{n}$.

(b) Express $\overrightarrow{NC}$ in terms of $\mathbf{m}$ and $\mathbf{n}$.



1C Diagonals, midpoints and extensions

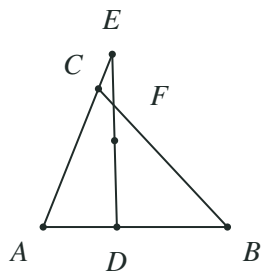
Q11

ABC is a triangle with $\overrightarrow{AB} = \mathbf{m}$ and $\overrightarrow{AC} = \mathbf{n}$.

D lies on AB with $AD : DB = 2 : 3$. AC is extended beyond C to E , where $CE = \frac{1}{4}AC$. F is the midpoint of DE .

(a) Express $\overrightarrow{BF}$ in terms of $\mathbf{m}$ and $\mathbf{n}$.

(b) Express $\overrightarrow{CF}$ in terms of $\mathbf{m}$ and $\mathbf{n}$.



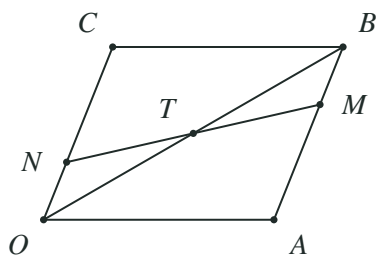
Q12

$OABC$ is a parallelogram with $\overrightarrow{OA} = \mathbf{m}$ and $\overrightarrow{OC} = \mathbf{n}$.

M lies on AB with $AM : MB = 2 : 1$. N lies on OC with $ON : NC = 1 : 2$. T is the midpoint of MN .

(a) Find $\overrightarrow{OT}$ in terms of $\mathbf{m}$ and $\mathbf{n}$.

(b) Use your result to decide whether T is the midpoint of OB . Explain your answer.



2A

**

Finding unknown coefficients

In Section 2, $\mathbf{m}$ and $\mathbf{n}$ have positive lengths and are not parallel, unless a question says otherwise. For equal vectors, the coefficient of $\mathbf{m}$ must match on both sides, and the coefficient of $\mathbf{n}$ must also match.

Example Problem

Find x and y if $x\mathbf{m} + (y - 1)\mathbf{n} = 3\mathbf{m} + 4\mathbf{n}$.

The idea

1. A coefficient is the number multiplying a vector. For example, the coefficient of $\mathbf{m}$ in $3\mathbf{m}$ is 3.
2. If equal vectors are written using the same two non-parallel vectors, compare their coefficients separately.

Worked Solution

$$x\mathbf{m} + (y - 1)\mathbf{n} = 3\mathbf{m} + 4\mathbf{n}$$

$$x = 3$$

$$y - 1 = 4$$

$$y = 5$$

$$x = 3, \quad y = 5$$

The two sides describe the same vector.

Compare the coefficients of $\mathbf{m}$.

Compare the coefficients of $\mathbf{n}$.

Add 1 to both sides of the second equation.

These values satisfy both equations at the same time.

2A Finding unknown coefficients

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Q13

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Find x and y if $(x - 2)m + (y + 3)n = 4m - 2n$.

Q14

**

Find p and q if $(2p + 1)m + (3 - q)n = 7m$.

Q15

**

$x(m + n) + y(m - n) = 7m + n$ and $y = 3$.
Find x .

Q16

**

For this question only, $n = 2m$.
 $xm + yn = 6m$.

(a) Find x when $y = 1$.

(b) Find x when $y = 2$.

2B

Forming simultaneous equations

Example Problem

Find x and y if $x(2\mathbf{m} - \mathbf{n}) + y(\mathbf{m} + 3\mathbf{n}) = 7\mathbf{m} + 7\mathbf{n}$.

The idea

1. Expand the brackets and collect all the multiples of $\mathbf{m}$ and all the multiples of $\mathbf{n}$.
2. Compare coefficients to obtain two simultaneous equations.
3. Solve the equations, then check both values in the original coefficients.

Worked Solution

$$x(2\mathbf{m} - \mathbf{n}) + y(\mathbf{m} + 3\mathbf{n}) = 7\mathbf{m} + 7\mathbf{n}$$

Start with the given vector equation.

$$2x\mathbf{m} - x\mathbf{n} + y\mathbf{m} + 3y\mathbf{n} = 7\mathbf{m} + 7\mathbf{n}$$

Expand both brackets.

$$(2x + y)\mathbf{m} + (-x + 3y)\mathbf{n} = 7\mathbf{m} + 7\mathbf{n}$$

Collect the terms multiplying each vector.

$$2x + y = 7$$

Compare the coefficients of $\mathbf{m}$.

$$-x + 3y = 7$$

Compare the coefficients of $\mathbf{n}$. We now have two simultaneous equations.

$$y = 7 - 2x$$

Rearrange the first equation to make y the subject.

$$-x + 3(7 - 2x) = 7$$

Substitute into the second equation.

$$-7x = -14$$

Expand and collect terms.

$$x = 2, \quad y = 3$$

Find x , then substitute into $y = 7 - 2x$.

$$2(2) + 3 = 7, \quad -2 + 3(3) = 7$$

Check both coefficients, not just one.

2B Forming simultaneous equations

Q17

Find x and y if $x(3m + n) + y(m - 2n) = 11m - n$.

Q18

Find x and y if $(x - 2y)m + (3x + y)n = -5m + 13n$.

2B Forming simultaneous equations

Q19

Find x and y if $\frac{x}{2}(m + 2n) - y(2m - n) = -m + 8n$.

Q20

Find x and y if $(2x - y)(m - n) + (x + y)(2m + n) = 10m - 7n$.

2C

Parallel vectors: length and direction

Example Problem

$u = (r + 1)m + 6n$ and $v = 2m + 3n$ are parallel.

- (a) Find r .
 (b) How many times as long as v is u ? Do they point in the same or opposite directions?

The idea

1. Parallel vectors are multiples of one another. Write $u = \lambda v$, where the Greek letter λ stands for an unknown number.
2. Compare the coefficients of m and n to find λ and r . The size of λ tells you the length factor; a negative sign reverses the direction.

Worked Solution

(a) $u = \lambda v$

Introduce the unknown scale factor.

$$(r + 1)m + 6n = \lambda(2m + 3n)$$

Substitute the given expressions.

$$(r + 1)m + 6n = 2\lambda m + 3\lambda n$$

Multiply both terms inside the brackets by λ .

$$r + 1 = 2\lambda$$

Compare the coefficients of m .

$$6 = 3\lambda$$

Compare the coefficients of n . These are simultaneous equations for r and λ .

$$\lambda = 2$$

Divide the second equation by 3.

$$r + 1 = 4$$

Substitute $\lambda = 2$ into the first equation.

$$r = 3$$

Subtract 1. The vectors have the same direction because λ is positive.

(b) $u = 2v$

Twice as long, in the same direction.

2C Parallel vectors: length and direction

Q21

$$\mathbf{u} = (p - 1)\mathbf{m} + 8\mathbf{n} \text{ and } \mathbf{v} = 3\mathbf{m} + 2\mathbf{n}.$$

- (a) Find p so that $\mathbf{u}$ and $\mathbf{v}$ are parallel.
- (b) Describe the length of $\mathbf{u}$ compared with $\mathbf{v}$, and state whether their directions are the same or opposite.

Q22

$$\mathbf{u} = 4\mathbf{m} + (q + 1)\mathbf{n} \text{ and } \mathbf{v} = -2\mathbf{m} + 3\mathbf{n}.$$

- (a) Find q so that $\mathbf{u}$ and $\mathbf{v}$ are parallel.
- (b) Describe the length of $\mathbf{u}$ compared with $\mathbf{v}$, and state whether their directions are the same or opposite.

2C Parallel vectors: length and direction

Q23

$$\mathbf{u} = (k + 2)\mathbf{m} + (2k - 1)\mathbf{n} \text{ and } \mathbf{v} = 2\mathbf{m} + 5\mathbf{n}.$$

- (a) Find k so that $\mathbf{u}$ and $\mathbf{v}$ are parallel.
- (b) Describe the length of $\mathbf{u}$ compared with $\mathbf{v}$, and state whether their directions are the same or opposite.

Q24

$$\mathbf{u} = (2t - 1)\mathbf{m} + (t + 2)\mathbf{n} \text{ and } \mathbf{v} = 3\mathbf{m} - \mathbf{n}.$$

- (a) Find t so that $\mathbf{u}$ and $\mathbf{v}$ are parallel.
- (b) Describe the length of $\mathbf{u}$ compared with $\mathbf{v}$, and state whether their directions are the same or opposite.
- (c) Find the scale factor λ such that $\mathbf{u} = \lambda\mathbf{v}$.

3A

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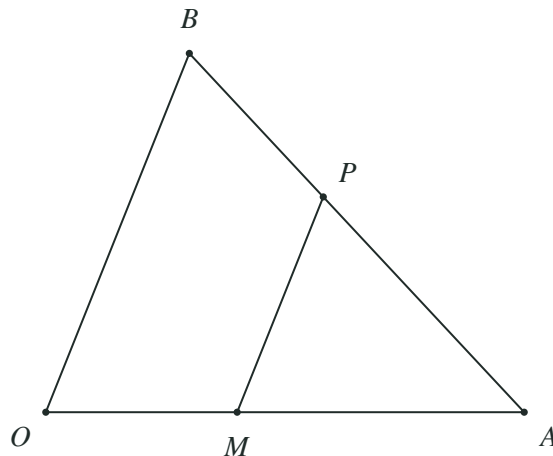
Finding ratios using parallel lines

Example Problem

OAB is a triangle with $\overrightarrow{OA} = \mathbf{m}$ and $\overrightarrow{OB} = \mathbf{n}$.

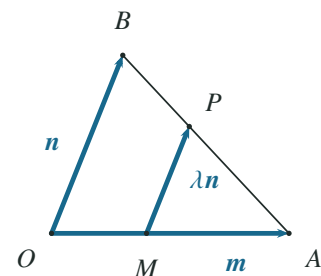
M lies on OA with $OM : MA = 2 : 3$. P lies on AB and MP is parallel to OB .

Use vectors to find $AP : PB$.



The idea

1. Write $\overrightarrow{AP} = t\overrightarrow{AB}$, where t is the unknown fraction of AB .
2. Use O to A to P to find $\overrightarrow{OP}$. Then subtract $\overrightarrow{OM}$ to find $\overrightarrow{MP}$.
3. Since MP is parallel to OB , write $\overrightarrow{MP} = \lambda\overrightarrow{OB}$. Compare the $\mathbf{n}$ coefficients to find t .
4. Convert the fraction t into the ratio $AP:PB$.



$$OM : MA = 2 : 3$$

3A

Finding ratios using parallel lines

Worked Solution

1. Express $\overrightarrow{OP}$ using the unknown fraction

$$\overrightarrow{AP} = t\overrightarrow{AB}$$

The number t tells us how far P is along AB .

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$

Find AB by travelling through O .

$$\overrightarrow{AB} = -\mathbf{m} + \mathbf{n}$$

AO is the reverse of OA .

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

Follow O to A to P .

$$\overrightarrow{OP} = \mathbf{m} + t\overrightarrow{AB}$$

Replace AP by t times AB .

$$\overrightarrow{OP} = \mathbf{m} + t(-\mathbf{m} + \mathbf{n})$$

Substitute the expression for AB .

$$\overrightarrow{OP} = (1 - t)\mathbf{m} + t\mathbf{n}$$

Expand and collect terms.

2. Express $\overrightarrow{MP}$

$$\overrightarrow{OM} = \frac{2}{5}\mathbf{m}$$

$OM : MA = 2 : 3$, so OM has two of the five equal parts of OA .

$$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$$

The route O to M to P reaches the same point P .

$$\overrightarrow{MP} = \overrightarrow{OP} - \overrightarrow{OM}$$

Subtract OM from both sides.

$$\overrightarrow{MP} = (1 - t)\mathbf{m} + t\mathbf{n} - \frac{2}{5}\mathbf{m}$$

Substitute the expressions already found.

$$\overrightarrow{MP} = \left(\frac{3}{5} - t\right)\mathbf{m} + t\mathbf{n}$$

Collect the multiples of $\mathbf{m}$.

3. Use the parallel lines and interpret t

$$\overrightarrow{MP} = \lambda\overrightarrow{OB}$$

MP is parallel to OB , so it is a multiple of OB .

$$\left(\frac{3}{5} - t\right)\mathbf{m} + t\mathbf{n} = 0\mathbf{m} + \lambda\mathbf{n}$$

$\overrightarrow{OB} = \mathbf{n}$. The right-hand side has no $\mathbf{m}$ term.

$$\frac{3}{5} - t = 0$$

Compare the coefficients of $\mathbf{m}$.

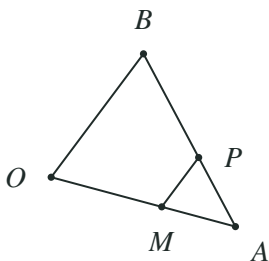
$$t = \frac{3}{5}$$

Add t to both sides. Comparing $\mathbf{n}$ coefficients also gives $\lambda = t$.

$$AP : PB = 3 : 2$$

AP is three fifths of AB , so PB is the remaining two fifths.

Use vectors to find $AP : PB$.



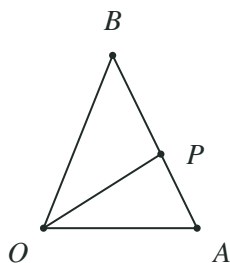
3A Finding ratios using parallel lines

Q26

OAB is a triangle with $\overrightarrow{OA} = 2\mathbf{m}$ and $\overrightarrow{OB} = 3\mathbf{n}$.

P lies on AB . $\overrightarrow{AP} = t\overrightarrow{AB}$ and $\overrightarrow{OP} = (1+s)\mathbf{m} + (2s+1)\mathbf{n}$.

Find s and t .



The diagram shows a parallelogram $OACB$ with origin O at the bottom-left vertex. Vertex A is at the bottom-right, and vertex B is at the top-right. Vertex C is at the top-left. The diagonal AB is drawn. Point M is the midpoint of the side CB . A line segment OM is drawn from the origin to M . Another line segment AC is drawn from vertex A to vertex C . The intersection of OM and AC is labeled X .

3A Finding ratios using parallel lines

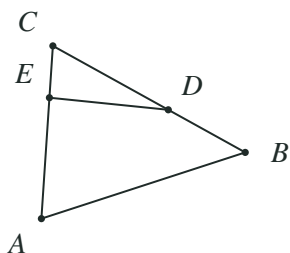
Q28

ABC is a triangle with $\overrightarrow{AB} = \mathbf{m}$ and $\overrightarrow{AC} = \mathbf{n}$.

D lies on BC with $BD : DC = 2 : 3$. E lies on AC and $\overrightarrow{DE} = k(2\mathbf{m} - \mathbf{n})$.

(a) Find k .

(b) Find $AE : EC$.



3B

★★★★

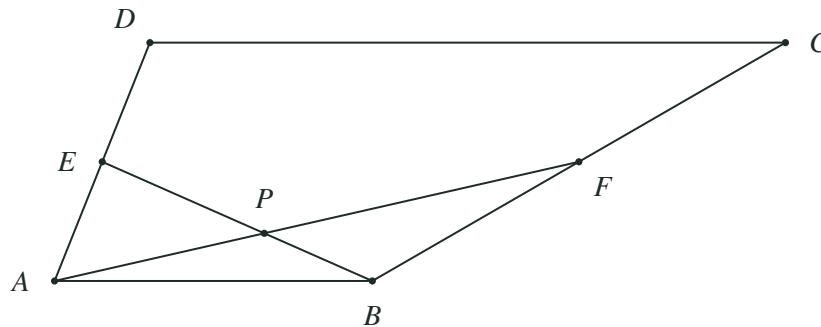
Finding where two lines intersect

Example Problem

$ABCD$ is a trapezium with $\overrightarrow{AB} = \mathbf{m}$, $\overrightarrow{AD} = \mathbf{n}$ and $\overrightarrow{DC} = 2\mathbf{m}$.

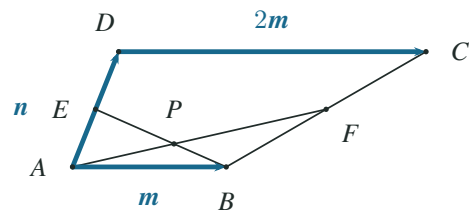
E is the midpoint of AD and F is the midpoint of BC . AF and BE intersect at P .

Find $AP : PF$.



The idea

1. Find AF and BE by combining the given vectors along routes.
2. Write $\overrightarrow{AP} = t\overrightarrow{AF}$ and $\overrightarrow{BP} = s\overrightarrow{BE}$. The two unknown numbers describe the position of the same point P on two different lines.
3. Express AP in two ways and compare coefficients. This gives two simultaneous equations in s and t .
4. Solve them and convert t into the requested ratio $AP:PF$.



$$AE : ED = 1 : 1$$

$$BF : FC = 1 : 1$$

3B

Finding where two lines intersect

Worked Solution

1. Find the two lines through P

$$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AD} + \overrightarrow{DC}$$

$$\overrightarrow{BC} = -\mathbf{m} + \mathbf{n} + 2\mathbf{m} = \mathbf{m} + \mathbf{n}$$

$$\overrightarrow{BF} = \frac{1}{2}(\mathbf{m} + \mathbf{n})$$

$$\overrightarrow{AF} = \overrightarrow{AB} + \overrightarrow{BF}$$

$$\overrightarrow{AF} = \mathbf{m} + \frac{1}{2}(\mathbf{m} + \mathbf{n})$$

$$\overrightarrow{AF} = \frac{3}{2}\mathbf{m} + \frac{1}{2}\mathbf{n}$$

$$\overrightarrow{AE} = \frac{1}{2}\mathbf{n}$$

$$\overrightarrow{BE} = -\mathbf{m} + \frac{1}{2}\mathbf{n}$$

Follow B to A to D to C .

Substitute the three given vectors and simplify.

F is the midpoint of BC .

Follow A to B to F .

Substitute the expression for BF .

Expand and collect terms.

E is the midpoint of AD .

Follow B to A to E .

2. Describe AP in two ways

$$\overrightarrow{AP} = t\overrightarrow{AF}$$

$$\overrightarrow{AP} = \frac{3t}{2}\mathbf{m} + \frac{t}{2}\mathbf{n}$$

$$\overrightarrow{BP} = s\overrightarrow{BE}$$

$$\overrightarrow{AP} = \overrightarrow{AB} + \overrightarrow{BP}$$

$$\overrightarrow{AP} = \mathbf{m} + s(-\mathbf{m} + \frac{1}{2}\mathbf{n})$$

$$\overrightarrow{AP} = (1-s)\mathbf{m} + \frac{s}{2}\mathbf{n}$$

Let t be the fraction of AF from A to P .

Multiply the expression for AF by t .

Use a separate unknown s for the fraction of BE from B to P .

Follow A to B to P .

Substitute the expression for BE .

Expand and collect terms.

3B

Finding where two lines intersect

Worked Solution - continued

3. Solve the simultaneous equations

$$\frac{3t}{2} = 1 - s$$

$$\frac{t}{2} = \frac{s}{2}$$

$$t = s$$

$$\frac{3t}{2} = 1 - t$$

$$3t = 2 - 2t$$

$$5t = 2, \quad t = \frac{2}{5}$$

$$AP : PF = 2 : 3$$

Compare the coefficients of m in the two expressions for AP .

Compare the coefficients of n .

Multiply the second equation by 2.

Substitute $s = t$ into the first equation.

Multiply both sides by 2.

Collect the terms in t , then divide by 5.

AP is two fifths of AF ; PF is the remaining three fifths.

3B Finding where two lines intersect

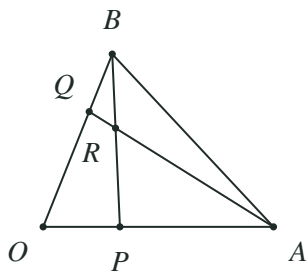
Q29

OAB is a triangle with $\overrightarrow{OA} = \mathbf{m}$ and $\overrightarrow{OB} = \mathbf{n}$.

P lies on OA with $OP : PA = 1 : 2$. Q lies on OB with $OQ : QB = 2 : 1$. The lines AQ and BP meet at R .

(a) Find $AR : RQ$.

(b) Find $BR : RP$.



3B Finding where two lines intersect

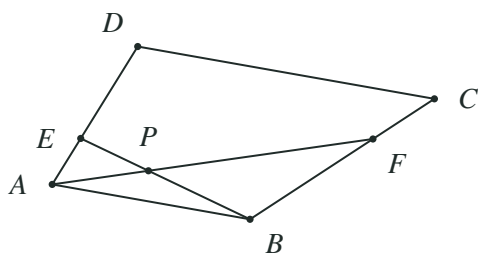
Q30

$ABCD$ is a trapezium with $\overrightarrow{AB} = \mathbf{m}$, $\overrightarrow{AD} = \mathbf{n}$ and $\overrightarrow{DC} = \frac{3}{2}\mathbf{m}$.

E lies on AD with $AE : ED = 1 : 2$. F lies on BC with $BF : FC = 2 : 1$. AF and BE intersect at P .

(a) Find $AP : PF$.

(b) Find $BP : PE$.



3B Finding where two lines intersect

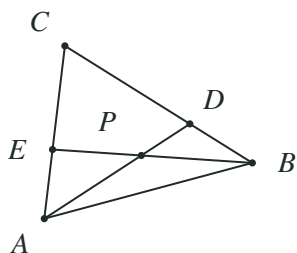
Q31

ABC is a triangle with $\overrightarrow{AB} = \mathbf{m}$ and $\overrightarrow{AC} = \mathbf{n}$.

D lies on BC with $BD : DC = 1 : 2$. E lies on AC with $AE : EC = 2 : 3$. AD and BE intersect at P .

(a) Find $\overrightarrow{AP}$ in terms of $\mathbf{m}$ and $\mathbf{n}$.

(b) Find $BP : PE$.



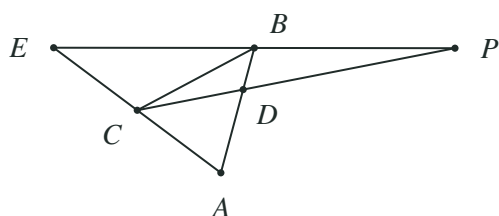
3B Finding where two lines intersect

Q32

ABC is a triangle with $\overrightarrow{AB} = \mathbf{m}$ and $\overrightarrow{AC} = \mathbf{n}$.

D lies on AB with $AD : DB = 2 : 1$. AC is extended beyond C to E with $CE = AC$. The straight lines CD and BE , extended as necessary, meet at P .

- (a) Find $\overrightarrow{AP}$ in terms of $\mathbf{m}$ and $\mathbf{n}$.
- (b) Find $BP : PE$.
- (c) Does P lie between B and E , beyond B , or beyond E ? Justify your answer using a vector relation from your working.



3C

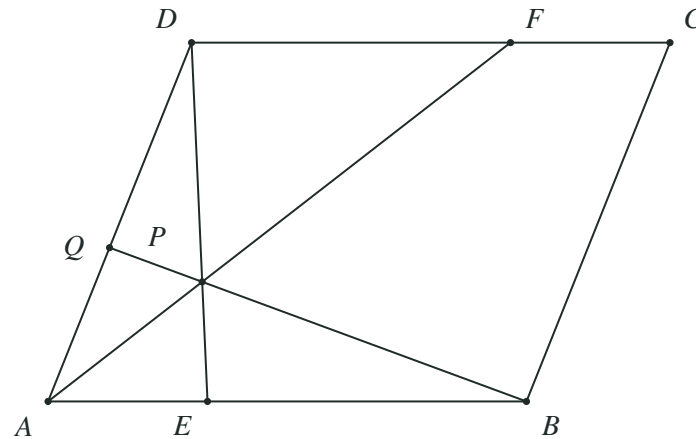
Finding a second intersection

Example Problem

$ABCD$ is a parallelogram with $\overrightarrow{AB} = \mathbf{m}$ and $\overrightarrow{AD} = \mathbf{n}$.

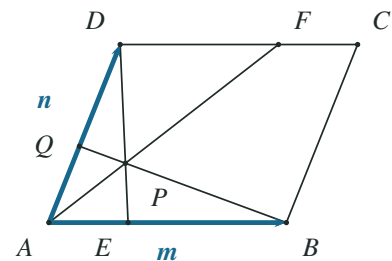
E lies on AB with $AE : EB = 1 : 2$. F lies on DC with $DF : FC = 2 : 1$. AF and DE intersect at P . The straight line BP meets AD at Q .

Find $AQ : QD$.



The idea

1. Use the intersection of AF and DE to find AP .
2. Use that result to express AQ by travelling A to B to Q .
3. Q lies on AD , so AQ has no $\mathbf{m}$ term. Use this to locate Q .
4. Convert the fraction of AD into $AQ:QD$.



$$AE : EB = 1 : 2$$

$$DF : FC = 2 : 1$$

3C

Finding a second intersection

Worked Solution

1. Locate P

$$\overrightarrow{AF} = \frac{2}{3}\mathbf{m} + \mathbf{n}$$

$$\overrightarrow{DE} = \frac{1}{3}\mathbf{m} - \mathbf{n}$$

$$\overrightarrow{AP} = t\overrightarrow{AF}$$

$$\overrightarrow{AP} = \frac{2t}{3}\mathbf{m} + t\mathbf{n}$$

$$\overrightarrow{DP} = s\overrightarrow{DE}$$

$$\overrightarrow{AP} = \mathbf{n} + s\left(\frac{1}{3}\mathbf{m} - \mathbf{n}\right)$$

$$\overrightarrow{AP} = \frac{s}{3}\mathbf{m} + (1-s)\mathbf{n}$$

$$\frac{2t}{3} = \frac{s}{3}, \quad t = 1 - s$$

$$s = 2t$$

$$t = 1 - 2t$$

$$t = \frac{1}{3}$$

$$\overrightarrow{AP} = \frac{2}{9}\mathbf{m} + \frac{1}{3}\mathbf{n}$$

Follow A to D to F. DF is two thirds of DC .

Follow D to A to E. AE is one third of AB .

Introduce the unknown fraction t along AF .

Multiply the expression for AF by t .

Use a second unknown for the position of P on DE .

Follow A to D to P.

Collect the multiples of each vector.

Comparing coefficients gives two simultaneous equations.

Multiply the first equation by 3.

Substitute $s = 2t$ into the second equation.

Solve $3t = 1$.

Substitute t into the expression for AP .

3C

Finding a second intersection

Worked Solution - continued

2. Use P to locate Q

$$\overrightarrow{BP} = \overrightarrow{BA} + \overrightarrow{AP}$$

$$\overrightarrow{BP} = -\frac{7}{9}\mathbf{m} + \frac{1}{3}\mathbf{n}$$

$$\overrightarrow{BQ} = u\overrightarrow{BP}$$

$$\overrightarrow{AQ} = \mathbf{m} + u\left(-\frac{7}{9}\mathbf{m} + \frac{1}{3}\mathbf{n}\right)$$

$$\overrightarrow{AQ} = \left(1 - \frac{7u}{9}\right)\mathbf{m} + \frac{u}{3}\mathbf{n}$$

$$1 - \frac{7u}{9} = 0$$

$$u = \frac{9}{7}$$

$$\overrightarrow{AQ} = \frac{3}{7}\mathbf{n}$$

$$AQ : QD = 3 : 4$$

Find the direction from B through P .

Subtract $\mathbf{m}$ from the expression for $\overrightarrow{AP}$.

Introduce a new number u to locate Q on the line BP .

Follow A to B to Q .

Expand and collect terms.

Q is on AD . AQ is a multiple of $\mathbf{n}$, with no $\mathbf{m}$ term.

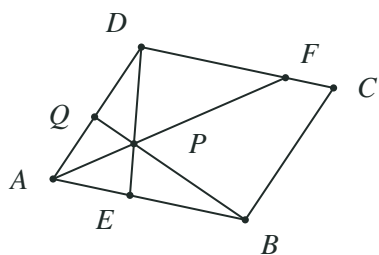
Solve for the multiplier along BP .

Substitute u into the remaining coefficient.

AQ is three sevenths of AD ; QD is the remaining four sevenths.

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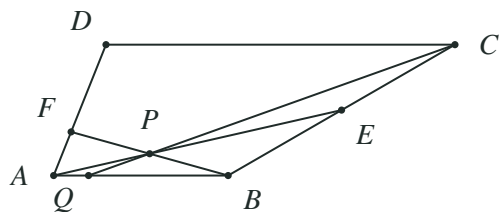
3C Finding a second intersection

Q34

$ABCD$ is a trapezium with $\overrightarrow{AB} = \mathbf{m}$, $\overrightarrow{AD} = \mathbf{n}$ and $\overrightarrow{DC} = 2\mathbf{m}$.

E is the midpoint of BC . F lies on AD with $AF : FD = 1 : 2$. AE and BF intersect at P . The straight line CP meets AB at Q .

Find $AQ : QB$.



3C Finding a second intersection

Q35

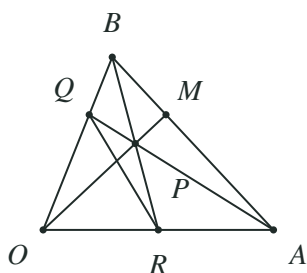
OAB is a triangle with $\overrightarrow{OA} = \mathbf{m}$ and $\overrightarrow{OB} = \mathbf{n}$.

M lies on AB with $AM : MB = 2 : 1$. P lies on OM with $OP : PM = 3 : 1$. AP meets OB at Q , and BP meets OA at R .

(a) Find $OQ : QB$.

(b) Find $OR : RA$.

(c) Is QR parallel to AB ? Justify your answer using vectors.

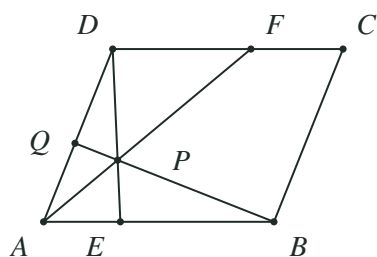


3C Finding a second intersection

Q36

$ABCD$ is a parallelogram with $\overrightarrow{AB} = \mathbf{m}$ and $\overrightarrow{AD} = \mathbf{n}$. E lies on AB with $AE : EB = 1 : 2$. F lies on DC with $DF : FC = r : 1$, where $r > 0$. AF and DE intersect at P . The straight line BP meets AD at Q .

- (a) Given $AQ : QD = 5 : 6$, find r .
- (b) Challenge: show that $AQ : QD = (r + 1) : 2r$ for every $r > 0$.



Solving a vector problem: tips and tricks

1. Draw or copy the diagram. Label the given vectors, ratios and parallel lines.
2. Find a route from the starting point of the vector you need to its finishing point. Add the vectors along it; reverse a sign when travelling backwards.
3. Use a letter for an unknown fraction along a line, for example $\overrightarrow{AP} = t\overrightarrow{AB}$.
4. Look for a second route to the same point. Equate the two expressions. For parallel vectors, write one as a multiple of the other.
5. When the two reference vectors are non-parallel, compare coefficients to form equations. Solve them, then convert any fractions into the requested ratios.