

GCSE MATHS

Iteration

Long workbook

Questions on GCSE Maths topics, including challenges beyond typical exam questions.

Contents

	Questions	Difficulty	Page
Topic recap			2
1. Using iterative formulae	Q1–6	**	3
2. Rearranging equations for iteration	Q7–12	***	6
3. Locating and rounding roots	Q13–18	***	10
4. Comparing iterative methods	Q19–24	****	12
Solving principles			15

Topic recap

Iteration

An iterative formula uses the previous value to calculate the next. In $x_{n+1} = g(x_n)$, x_0 is the starting value.

Calculator accuracy

Keep unrounded values in the calculator between iterations. Round only the values you report.

A limiting value

A sequence converges when its values approach a single limit L . In these iterative formulae the limit satisfies $L = g(L)$.

Changes of sign

Opposite signs on an unbroken graph show that it crosses the x -axis between the two values tested. This gives another way to locate a root.

A calculator is needed for this pack.

1. Using iterative formulae

Example Problem 1

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Use $x_{n+1} = \sqrt{12 + x_n}$, with $x_0 = 3$, to find x_1, x_2 to 4 decimal places.

The idea

1. Store the full answer after each calculation and reuse it.

Worked Solution

$$x_1 = \sqrt{15} = 3.872983346 \dots$$

Put 3 into the formula.

$$x_2 = \sqrt{12 + \sqrt{15}} = 3.9840913 \dots$$

Use the stored value of x_1 .

$$x_1 = 3.8730$$

Round the reported values.

$$x_2 = 3.9841$$

1 Using iterative formulae

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Q1 Use $x_{n+1} = \frac{x_n+6}{2}$ with $x_0 = 2$ to calculate x_1, x_2, x_3 . Give all three values.

**

Q2 Use $x_{n+1} = \sqrt{10 + x_n}$ with $x_0 = 3$ to calculate x_1, x_2, x_3 . Give non-integer values to 4 decimal places.

**

Q3 Use $x_{n+1} = 1 + \frac{6}{x_n}$ with $x_0 = 2$ to calculate x_1, x_2, x_3 . Give all three values.

**

Q4 $x_{n+1} = 2x_n - 3$ and $x_3 = 13$. Find x_0 .

**

Q5 $x_{n+1} = \frac{1}{2}x_n + c$, where c is constant. Given $x_0 = 4$ and $x_1 = 7$, find x_2 .

**

Q6 Use $x_{n+1} = \sqrt{x_n + 8}$ with $x_0 = 2$ to find x_2 to 4 decimal places.

**

2. Rearranging equations for iteration

Example Problem 2

Rearrange $x^2 + x - 8 = 0$ into $x = \sqrt{8 - x}$ for positive x .

The idea

1. Rearrange to make x the subject on one side, while allowing x on the other.

Worked Solution

$$x^2 = 8 - x$$

Subtract x and add 8.

$$x = \sqrt{8 - x}$$

Take the positive square root.

$$x_{n+1} = \sqrt{8 - x_n}$$

Use the previous value on the right to generate the next.

2. Rearranging equations for iteration

Example Problem 3

$x_{n+1} = \sqrt{20 + x_n}$ converges to a positive limit. Find it.

The idea

1. Both successive values approach L , so replace each with L and solve the resulting equation.

Worked Solution

$$L = \sqrt{20 + L}$$

Use the same limit on both sides.

$$L^2 - L - 20 = 0$$

Square and rearrange.

$$(L - 5)(L + 4) = 0$$

Factorise.

$$L = 5$$

The positive limit is 5.

2 Rearranging equations for iteration

Q7 Show how $x^2 - x - 12 = 0$ gives the iterative formula $x_{n+1} = \sqrt{x_n + 12}$. Use positive x .

Q8 Show how $x^3 + x - 7 = 0$ gives the iterative formula $x_{n+1} = \sqrt[3]{7 - x_n}$. Use positive x .

Q9 Show how $x^3 + 2x - 10 = 0$ gives the iterative formula $x_{n+1} = \frac{10}{x_n^2 + 2}$. Use positive x .

Q10 Show how $x^2 - 5 = 0$ gives the iterative formula $x_{n+1} = \frac{1}{2} \left(x_n + \frac{5}{x_n} \right)$. Use positive x .

Q11 $x_{n+1} = \sqrt{6 + x_n}$ converges to a positive limit. Find the limit exactly.

Q12 Use $x_{n+1} = \frac{12}{x_n}$ with $x_0 = 2$. (a) Find x_1, x_2, x_3, x_4 . (b) Does this sequence approach a single value?

3. Locating and rounding roots

Example Problem 4

Show a root of $f(x) = x^3 - 10$ is 2.15 to 2 decimal places.

The idea

1. Use the boundaries for rounding to 2.15: 2.145 and 2.155.

Worked Solution

$$f(2.145) = -0.130801375$$

Test the lower rounding boundary.

$$f(2.155) = 0.007873875$$

Test the upper rounding boundary.

$$2.145 < r < 2.155$$

The cubic changes sign, so its graph crosses the x -axis between these values.

$$r = 2.15 \quad (2 \text{ d.p.})$$

All values in the interval round to 2.15.

3 Locating and rounding roots

Q13 Show that $f(x) = x^3 + x - 5$ has a root between 1 and 2.

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Q14 Show that $f(x) = x^3 - 2x - 4$ has a root between 1.5 and 2.5.

**

Q15 For $f(x) = x^3 + x - 5$, evaluate f at 1.51 and 1.52. Use these values to give a root to 1 decimal place.

Q16 The graph of $y = f(x)$ has no breaks between $x = 2$ and $x = 3$ and crosses the x -axis once, at $x = r$. Given $f(2.345) < 0$ and $f(2.355) > 0$, give r to 2 decimal places.

Q17 An equation has a root between 1.234 and 1.236. Is this enough to state its value to 2 decimal places? Explain.

Q18 Use $x_{n+1} = \sqrt{5 - x_n}$ with $x_0 = 2$ to find x_4 to 4 decimal places.

4. Comparing iterative methods

Example Problem 5

$x_{n+1} = \frac{1}{2}(x_n + 16/x_n)$ converges to a positive limit. Find it.

The idea

1. At a limit, the next value and previous value approach the same number.

Worked Solution

$$L = \frac{1}{2}(L + 16/L)$$

Replace both successive values by L .

$$2L^2 = L^2 + 16$$

Multiply by $2L$.

$$L^2 = 16 \Rightarrow L = 4$$

Choose the positive root.

4 Comparing iterative methods

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Q19 The iteration $x_{n+1} = \frac{1}{2}(x_n + 9/x_n)$ has $x_0 = 6$. Calculate x_1, x_2, x_3 to 4 decimal places.

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Q20 For $x^2 = 2$, method A uses $x_{n+1} = 2/x_n$; method B uses $x_{n+1} = (x_n + 2/x_n)/2$. Use $x_0 = 1$ for both methods. Calculate x_1, x_2, x_3 for each, giving non-integer values to 4 decimal places, and say which appears to get closer to a solution of $x^2 = 2$.

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Q21 $x_{n+1} = 3x_n - 4$. Find x_0 so that every term has the same value.

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Q22 Use $x_{n+1} = \frac{1}{2} \left(x_n + \frac{7}{x_n} \right)$ with $x_0 = 3$ to find the positive solution of $x^2 = 7$ to 3 decimal places. Show your iterations.

Q23 $x_{n+1} = ax_n + b$. Given $x_0 = 1$, $x_1 = 3$ and $x_2 = 4$: (a) find a and b ; (b) find the limit if the iteration converges.

Q24 A positive root of $x^2 + x - 3 = 0$ is estimated by iteration. Show that it is 1.303 to 3 decimal places by evaluating the quadratic at suitable rounding boundaries.

Solving principles

1. Substitute the latest value

Every new term uses the value immediately before it.

2. Check the original equation

A rearrangement can have restrictions or additional roots.

3. Report the numerical solution

Show the iterations and round the settled value to the accuracy requested. If asked to check a root using a change of sign, evaluate the original expression on either side.

4. Choose a useful rearrangement

Equivalent equations can produce iterations with different behaviour.