

## GCSE MATHS

# Functions

Long workbook

Questions on GCSE Maths topics, including challenges beyond typical exam questions.

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# Topic recap

## Function notation

$f(x) = 3x - 2$  means multiply the input by 3, then subtract 2. Thus  $f(4) = 10$ ;  $f(x)$  does not mean  $f$  times  $x$ .

## Composition

$fg(x)$  means  $f(g(x))$ : apply  $g$  first, then  $f$ . The order usually changes the answer.

## Inverses

$f^{-1}$  reverses  $f$  on its allowed inputs. It does not mean  $1/f$ . Write  $y = f(x)$  and rearrange for  $x$ .

## Restrictions

A denominator cannot be zero. For a square function an input restriction may be needed:  $f(x) = x^2, x \geq 0$ , has inverse  $\sqrt{x}$ . An inverse square-root formula accepts only values that make the expression inside the root non-negative.

# 1. Function notation and inputs

## Example Problem 1

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$f(x) = x^2 + 2$ . Find  $f(-3)$  and simplify  $f(x + 2)$ .

### The idea

1. Every appearance of the input is replaced together.
2. Use brackets before squaring an expression.

### Worked Solution

$$f(-3) = (-3)^2 + 2 = 11$$

Bracket the negative input.

$$f(x + 2) = (x + 2)^2 + 2$$

Replace the entire input.

$$f(x + 2) = x^2 + 4x + 6$$

Expand the square.

**1 Function notation and inputs**

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**Q1**  $f(x) = 4x - 7$ . Find  $f(5)$ .

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**Q2**  $g(x) = x^2 + 3x$ . Find  $g(-4)$ .

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**Q3**  $h(x) = 5 - 2x$ . Find the input that gives an output of 17.

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**Q4**  $f(x) = x^2 - 4$ . Find all inputs giving output 21.

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**Q5**  $f(x) = 3x + 2$ . Simplify  $f(2x - 1)$ .

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**Q6**  $f(x) = 2x^2 - 1$ . Simplify  $f(x + 1) - f(x)$ .

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## 2. Composite functions

### Example Problem 2

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$f(x) = 4x + 1$  and  $g(x) = x - 2$ . Find  $fg(x)$  and  $gf(x)$ .

### The idea

1. Start with the function nearest  $x$ .
2. Write a separate expression for each order.

### Worked Solution

$$fg(x) = 4(x - 2) + 1 = 4x - 7$$

Apply  $g$  then  $f$ .

$$gf(x) = (4x + 1) - 2 = 4x - 1$$

Apply  $f$  then  $g$ .

**2 Composite functions**

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**Q7**  $f(x) = 2x + 3$  and  $g(x) = x^2$ .

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**(a)** Find  $fg(4)$ .**(b)** Find  $gf(4)$ .**Q8**  $f(x) = 3x - 2$  and  $g(x) = 5x + 1$ . Simplify  $fg(x)$ .

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**Q9**  $f(x) = x^2 + 1$  and  $g(x) = x - 3$ . Simplify  $gf(x)$ .

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**Q10**  $f(x) = 2x - 1$  and  $g(x) = x^2$ . Solve  $fg(x) = 17$ .

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**Q11**  $f(x) = x + 2$  and  $g(x) = 3x$ . Determine whether any values of  $x$  satisfy  $fg(x) = gf(x)$ .

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### 3. Inverse functions

#### Example Problem 3

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$f(x) = x^2 + 1$  for  $x \geq 0$ . Find  $f^{-1}(x)$ .

#### The idea

1. Rearrange for the original input.
2. Use the original restriction to select the correct root.

#### Worked Solution

$$y = x^2 + 1$$

$$x = \sqrt{y - 1}$$

$$f^{-1}(x) = \sqrt{x - 1}$$
$$x \geq 1$$

Write the output as  $y$ .

Use the condition  $x \geq 0$  to choose the non-negative root.

The expression inside the square root must be non-negative.

**3 Inverse functions**

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**Q12**  $f(x) = 5x - 4$ . Find  $f^{-1}(x)$ .

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**Q13**  $g(x) = \frac{x+7}{3}$ . Find  $g^{-1}(x)$ .

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**Q14**  $f(x) = 7 - 2x$ . Find  $f^{-1}(9)$ .

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**Q15**  $f(x) = x^2 + 4$  for  $x \geq 0$ . (a) Find  $f^{-1}(x)$ . (b) State which values of  $x$  can be used in this inverse formula.

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**Q16**  $g(x) = \frac{6}{x-2}$ ,  $x \neq 2$ . Find  $g^{-1}(x)$  and state the value of  $x$  that cannot be used in it.

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**Q17**  $f(x) = 3x + 1$  and  $g(x) = 2x - 5$ . Find  $(fg)^{-1}(x)$ .

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## 4. Equations and function reasoning

### Example Problem 4

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$f(x) = ax + b$ . Given  $f(1) = 7$  and  $f(3) = 15$ , find  $a, b$ .

### The idea

1. Each known pair gives one equation.
2. Solve the two equations together.

### Worked Solution

$$a + b = 7$$

$$3a + b = 15$$

$$2a = 8 \Rightarrow a = 4$$

$$b = 7 - 4 = 3$$

Substitute the two known pairs.

Subtract the equations.

Substitute  $a = 4$ .

## 4. Equations and function reasoning

### Example Problem 5

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$f(x) = ax + b$ , where  $a \neq 0$ . Find  $a$  and  $b$  if  $ff(x) = 4x + 9$  for every real  $x$ .

### The idea

1. Compose the function using brackets.
2. An identity must have the same coefficient of  $x$  and the same constant on both sides.

### Worked Solution

$$ff(x) = a(ax + b) + b = a^2x + b(a + 1)$$

Substitute  $ax+b$  as the input to  $f$ .

$$a^2 = 4$$

Match the coefficient and constant separately.

$$b(a + 1) = 9$$

$$a = 2 \quad \text{or} \quad a = -2$$

Keep both square roots.

$$a = 2 : \quad 3b = 9 \Rightarrow b = 3$$

Find  $b$  for the positive value of  $a$ .

$$a = -2 : \quad -b = 9 \Rightarrow b = -9$$

Find  $b$  for the negative value of  $a$ .

**4 Equations and function reasoning**

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**Q18**  $f(x) = x^2$  and  $g(x) = 2x + 3$ . Solve  $fg(x) = gf(x)$ .

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**Q19**  $f(x) = ax + b$ . Given  $f(2) = 9$  and  $f(5) = 21$ , find  $a$  and  $b$ .

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**Q20**  $f(x) = 2x + k$ . Given  $ff(3) = 27$ , find  $k$ .

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**Q21**  $f(x) = ax + b$ , where  $a \neq 0$ . Find all values of  $a, b$  for which  $ff(x) = x$  for every real  $x$ .

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**Q22**  $f(x) = \frac{1}{x-1}$  and  $g(x) = 2x$ . Find  $fg(x)$  and state the value of  $x$  that cannot be used in it.

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**Q23**  $f(x) = x^2 - 6x + 13$  for  $x \geq 3$ . (a) Find  $f^{-1}(x)$ . (b) State which values of  $x$  can be used in this inverse formula.

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# Solving principles

## 1. Replace the whole input

For  $f(x) = x^2 - 1$ ,  $f(x + 3) = (x + 3)^2 - 1$ ; use brackets.

## 2. Read composition from the inside

Track each output as the next input.

## 3. Check inverses by composition

An inverse should give  $f^{-1}(f(x)) = x$  on the allowed inputs.

## 4. Check every solution

Substitute into the original functions and respect any input restrictions.