

GCSE MATHS

Algebraic proof

Worked Solutions

Q1

★

$$2n$$

Even integers differ by two.

$$2n + 2$$

$$2m + 1$$

Odd integers also differ by two.

$$2m + 3$$

$$2m + 5$$

Q2

★★

$$a = 2m + 1$$

Let m and n be integers.

$$b = 2n + 1$$

$$a + b = 2m + 2n + 2 = 2(m + n + 1)$$

Factor out two.

$$m + n + 1 \text{ is an integer}$$

The sum is twice an integer, so is even.

Q3

★★

$$(2m + 1)(2n + 1) = 4mn + 2m + 2n + 1$$

Use integer m, n and expand.

$$= 2(2mn + m + n) + 1$$

Collect a factor of two.

$$2mn + m + n \text{ is an integer}$$

The product has the form $2k + 1$.

Q4

★★

$$(2n + 1)^2 = 4n^2 + 4n + 1$$

Let n be an integer and expand.

$$= 2(2n^2 + 2n) + 1$$

This is one more than twice an integer.

Q5

★★

$$a = 2m$$

Let m, n be integers.

$$b = n$$

$$ab = 2mn = 2(mn)$$

The integer mn is multiplied by two.

Q6

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$$\begin{aligned}
 & n \\
 & n + 1 \\
 & n + 2 \\
 & n + (n + 1) + (n + 2) = 3n + 3 = 3(n + 1)
 \end{aligned}$$

Write the consecutive integers using an integer n .

The sum is three times an integer.

Q7

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$$\begin{aligned}
 & n + (n + 1) + (n + 2) = 3n + 3 \\
 & = 3(n + 1)
 \end{aligned}$$

Let n be an integer and add.

The sum is three times an integer.

Q8

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$$\begin{aligned}
 & \text{sum} = 4n + 6 \\
 & 4n + 6 = 2(2n + 3)
 \end{aligned}$$

Use an integer n for the first value.

The remaining factor is an integer.

Q9

$$\begin{aligned}
 & n \text{ even} \Rightarrow n(n + 1) \text{ even} \\
 & n \text{ odd} \Rightarrow n + 1 \text{ even} \\
 & n(n + 1) \text{ is even in both cases}
 \end{aligned}$$

An even factor makes an even product.

An integer must be either odd or even.

The argument covers every integer n .

Q10

$$n, n + 1, n + 2 \text{ contain a multiple of } 3$$

Three consecutive integers cover all three remainders.

$$\text{At least one factor is even.}$$

Any two consecutive integers include an even one.

$$\text{The product is divisible by } 6$$

The product is divisible by both 2 and 3, so it is divisible by 6.

Q11

$$\begin{aligned}(n+1)^2 - n^2 &= n^2 + 2n + 1 - n^2 \\ &= 2n + 1\end{aligned}$$

Expand the larger square.

This is odd for every integer n .

Q12

$$\begin{aligned}(2n+3)^2 - (2n+1)^2 \\ &= 4n^2 + 12n + 9 - (4n^2 + 4n + 1) \\ &= 8n + 8 = 8(n+1)\end{aligned}$$

Let n be an integer.

Expand both complete squares.

The remaining factor is an integer.

Q13

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$$\begin{aligned}(x+3)^2 - (x-3)^2 \\ &= x^2 + 6x + 9 - x^2 + 6x - 9 = 12x\end{aligned}$$

Start from the left side.

Subtract every term in the second bracket.

Q14

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$$\begin{aligned}(x+2)(x-2) &= x^2 - 4 \\ x^2 - 4 + 4 &= x^2\end{aligned}$$

Use difference of two squares.

Add four.

Q15

$$\begin{aligned}\bar{x} &= \frac{3n+3}{3} \\ \bar{x} &= n+1\end{aligned}$$

Write the total divided by three.

This is the middle of the three integers.

Q16

$$\begin{aligned}n^2 + (n+1)^2 &= 2n^2 + 2n + 1 \\ &= 2(n^2 + n) + 1\end{aligned}$$

Expand and collect.

This is one more than an even integer.

Q17

$$n^2 + n = n(n + 1)$$

One of $n, n + 1$ is even.

$n(n + 1)$ is even.

Factor out n .

Of two consecutive integers, one is odd and the other is even.

An even factor makes the product even.

Q18

$$n = 2k + 1 \Rightarrow n^2 - 1 = 4k(k + 1)$$

$$k(k + 1) = 2m \text{ for an integer } m$$

$$n^2 - 1 = 8m$$

Let k be an integer and expand.

One of the consecutive factors is even.

Therefore 8 divides the expression.

Q19

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$$4^2 + 4 + 1 = 21$$

$$21 = 3(7)$$

Choose a permitted input.

It has factors other than 1 and itself, so it is not prime.

Q20

$$0^2 = 0$$

$$1^2 = 1$$

Equality is not strictly greater.

Both are integer inputs.

Either example disproves the claim.

Q21

Let n be the middle integer.

$$\text{sum} = 5n$$

Use symmetric expressions.

The positive and negative offsets cancel.

Q22

$$n^3 - n = n(n^2 - 1) = (n - 1)n(n + 1)$$

One factor is a multiple of 3.

At least one factor is even.

Factor fully.

Every three consecutive integers contain one.

The product is divisible by both 2 and 3, hence by 6.

Q23

$$2n > 0 \Rightarrow n^2 < n^2 + 2n$$

$$n^2 + 2n = (n + 1)^2 - 1 < (n + 1)^2$$

No square number lies strictly between them.

Use positivity of n .

Compare with the next square number.

Therefore the expression is not a perfect square.

Q24

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1 and 5 are odd and differ by 4

$$5^2 - 1^2 = 24$$

24 is not divisible by 16

Test the stated conditions.

Find the difference.

This counterexample disproves the proposed claim.