

GCSE MATHS

Algebraic proof

Long workbook

Questions on GCSE Maths topics, including challenges beyond typical exam questions.

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Topic recap

Represent odd and even integers

Write an even integer as $2n$ and an odd integer as $2n + 1$, where n is an integer. Use different letters for unrelated integers.

Divisibility

To prove divisibility by k , write the expression as k times an integer. One of two consecutive integers is even.

Identities

An identity holds for every allowed value. Expand or factorise to connect the two expressions; checking a few substitutions is not a proof.

Disproving a claim

One counterexample is enough to disprove a statement. Give a value for which the statement is false.

1. Odd and even integers

Example Problem 1

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Prove that the sum of an even integer and an odd integer is odd.

The idea

1. Represent the integers independently.
2. Rearrange the sum into the form $2k + 1$.

Worked Solution

$$2m + (2n + 1) = 2(m + n) + 1$$

$m + n$ is an integer

Let m and n be any integers.

The sum is one more than twice an integer.

1 Odd and even integers

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Q1 Write expressions for two consecutive even integers and for three consecutive odd integers.

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Q2 Prove that the sum of two odd integers is even.

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Q3 Prove that the product of two odd integers is odd.

**

Q4 Prove that the square of an odd integer is odd.

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Q5 Prove that an even integer multiplied by any integer is even.

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Q6 Prove that the sum of three consecutive integers is a multiple of 3.

**

2. Consecutive numbers and divisibility

Example Problem 2

Prove that the product of consecutive integers is even.

The idea

1. Split into even and odd cases.
2. Identify an explicit factor of two in each case.

Worked Solution

$$n = 2k \Rightarrow n(n+1) = 2k(2k+1)$$

If the first integer is even, the product is even.

$$n = 2k+1 \Rightarrow n(n+1) = (2k+1)2(k+1)$$

If the first is odd, the next is even.

Both cases give an even product.

The two cases cover every integer n .

2 Consecutive numbers and divisibility

Q7 Prove that the sum of three consecutive integers is divisible by 3.

**

Q8 Prove that the sum of four consecutive integers is always even.

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Q9 Prove that the product of two consecutive integers is even.

Q10 Prove that the product of three consecutive integers is divisible by 6.

Q11 Prove that the difference between the squares of consecutive integers is odd.

Q12 Prove that the difference between the squares of consecutive odd integers is divisible by 8.

3. Identities and general arguments

Example Problem 3

Show that $(x + 4)^2 - (x - 4)^2 = 16x$.

The idea

1. Start with one side of the proposed identity.
2. Show algebraically that it becomes the other side.

Worked Solution

$$x^2 + 8x + 16 - (x^2 - 8x + 16)$$

Expand both squares.

$$x^2 + 8x + 16 - x^2 + 8x - 16 = 16x$$

Distribute the subtraction and collect.

3 Identities and general arguments

★★★★

Q13 Show that $(x + 3)^2 - (x - 3)^2 = 12x$ for every real x .

★★

Q14 Show that $(x + 2)(x - 2) + 4 = x^2$.

★★

Q15 Prove that the mean of three consecutive integers is the middle integer.

★★★

Q16 Prove that the sum of the squares of two consecutive integers is always odd.

Q17 For integer n , prove that $n^2 + n$ is even.

Q18 For an odd integer n , prove that $n^2 - 1$ is divisible by 8.

4. Proofs and counterexamples

Example Problem 4

**

Disprove the statement: $n^2 + 3$ is prime for every positive integer n .

The idea

1. One valid counterexample is enough.
2. Show both that your input is allowed and that the output fails the claim.

Worked Solution

$$n = 1 \Rightarrow n^2 + 3 = 4$$

Choose an allowed input.

$$4 = 2(2)$$

The output is composite.

4. Proofs and counterexamples

Example Problem 5

Prove that $n^2 + n$ is not a perfect square for any positive integer n .

The idea

1. Find the consecutive squares on either side of the expression.
2. Prove both inequalities for every positive integer n .

Worked Solution

$$n^2 < n^2 + n$$

Adding positive n gives a value greater than n^2 .

$$(n+1)^2 - (n^2 + n) = n + 1 > 0$$

The next square is greater than the expression.

$$n^2 < n^2 + n < (n+1)^2$$

Combine the two inequalities.

$n^2 + n$ is not a perfect square.

There is no square number strictly between consecutive squares.

4 Proofs and counterexamples

★★★★

Q19 Disprove the statement: $n^2 + n + 1$ is prime for every positive integer n .

★★

Q20 Give two different counterexamples to the statement: the square of an integer is always greater than the integer.

★★★

Q21 Prove that the sum of five consecutive integers is divisible by 5.

★★★

Q22 Prove that $n^3 - n$ is divisible by 6 for every integer n .

★★★★

Q23 Prove that for every positive integer n , $n^2 + 2n$ is not a perfect square.

★★★★

Q24 Two odd integers differ by 4. Is the difference of their squares always divisible by 16? Justify your answer.

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Solving principles

1. State what the letter represents

Include whether it is an integer, positive integer or another allowed value.

2. Keep the argument general

Numerical examples may suggest a pattern, but a proof must cover every allowed case.

3. Name the integer factor

After factorising, explain why the factor left over is an integer.

4. Check the wording of the claim

Always, sometimes and never require different kinds of argument. A counterexample must satisfy the stated conditions.